MA2108 Mathematical Analysis I

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References

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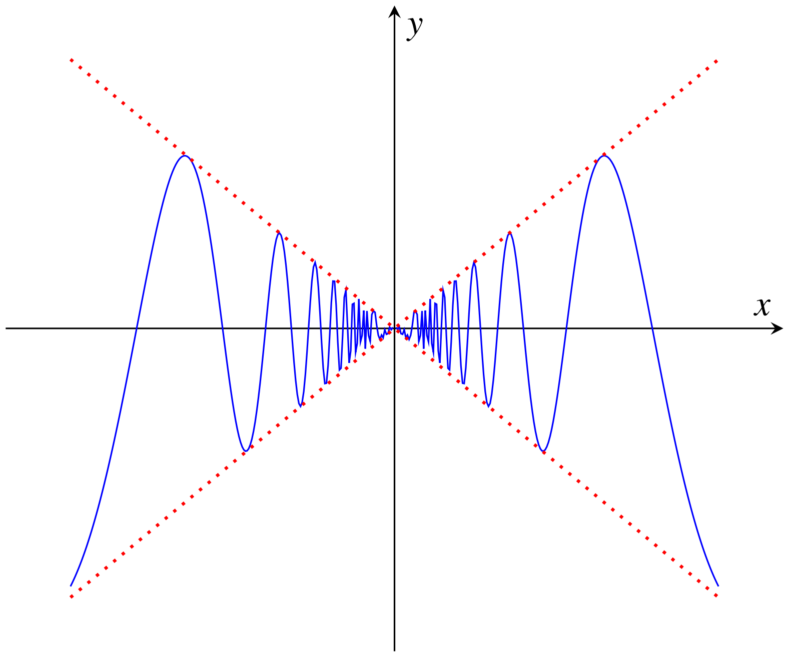

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Chapter 1
The Real Numbers, $\mathbb{R}$

1.1 Fields

We say that $\mathbb{R}$ is a complete ordered field. There are three big ideas to be discussed — completeness, ordering, and fields! We will first discuss the property on fields, and we say that $\mathbb{R}$ satisfies the field axioms (Definition 1.1)[†]. There are many properties which might be deemed *trivial* but we will still discuss them. For example, the trichotomy property of $\mathbb{R}$[‡] states that

$$\text{if } a,b\in\mathbb{R} \quad \text{then} \quad \text{either } a<b, a>b \text{ or } a=b.$$

This is intuitive!

Definition 1.1 (field axioms). A field consists of a set F satisfying the following properties:

(i) an additive map

$$+:F\times F\to F \quad \text{where} \quad (x,y)\mapsto x+y$$

(ii) the existence of an additive identity $0\in F$

(iii) a negation map

$$-:F\times F\to F \quad \text{where} \quad x\mapsto -x$$

(iv) a multiplication map

$$\cdot:F\times F\to F \quad \text{where} \quad (x,y)\mapsto xy$$

(v) the existence of a multiplicative identity $1\in F$

(vi) a reciprocal map

$$(-)^{-1}:F\setminus\{0\}\to F\setminus\{0\} \quad \text{where} \quad x\mapsto x^{-1}$$

such that the following properties are satisfied:

(i) $+$ is commutative, i.e. for all $x,y\in F$, we have $x+y=y+x$

(ii) $+$ is associative, i.e. for all $x,y,z\in F$, we have $(x+y)+z=x+(y+z)$

(iii) 0 is the identity for $+$, i.e. for all $x\in F$, we have $x+0=x=0+x$

(iv) $-$ is the additive inverse of addition, i.e. for all $x\in F$, we have $x+(-x)=0=(-x)+x$

(v) $\cdot$ is commutative, i.e. for all $x,y\in F$, we have $xy=yx$

(vi) $\cdot$ is associative, i.e. for all $x,y,z\in F$, we have $(xy)z=x(yz)$

(vii) 1 is the identity for $\cdot$, i.e. for all $x\in F$, we have $x1=x=1x$

(viii) $(-1)^{-1}$ is the inverse of $\cdot$, i.e. for all $x\in F$, we have $xx^{-1}=1=x^{-1}x$

(ix) $1\neq 0$, i.e. F is not the zero (trivial) field

(x) $\cdot$ is distributive over $+$, i.e. for all $x,y,z\in F$, we have

$$x(y+z)=xy+xz \quad \text{and} \quad (x+y)z=xz+yz$$

[†] An abrupt introduction.

[‡] In fact, we can regard the trichotomy property of $\mathbb{R}$ as a combination of the reflexivity and antisymmetry properties in Definition 1.2 and the comparability property in Definition 1.3. Alternatively, one can refer to **(iii)** of Proposition 1.5.

Remark 1.1. When we were discussing the properties of a field in Definition 1.1, recall that multiplication is denoted by $\cdot$, and we can *condense* $x \cdot y$ as xy. For example, refer to **(v)**, which can also be written as $x \cdot y = y \cdot x$.

Example 1.1. The best known fields are those of

$$\begin{aligned} \mathbb{Q} &= \text{field of rational numbers} \\ \mathbb{R} &= \text{field of real numbers} \\ \mathbb{C} &= \text{field of complex numbers} \end{aligned}$$

Example 1.2. In Number Theory or Abstract Algebra in general,

$$\begin{aligned} \mathbb{Q}_p &= \text{field of } p\text{-adic numbers} \\ \mathbb{F}_p &= \text{finite field of } p \text{ elements} \end{aligned}$$

Example 1.3. Let k be a field. Then, define

$$K(t) \quad \text{to be the field of rational functions over } K.$$

We then discuss the general properties of fields.

Proposition 1.1. The axioms for addition in Definition 1.1 imply the following statements: for all $x, y, z \in F$,

- **(i) Cancellation for $+$:** if $x + y = x + z$, then $y = z$;
- **(ii) Uniqueness of 0:** if $x + y = x$, then $y = 0$;
- **(iii) Uniqueness of negative:** if $x + y = 0$, then $y = -x$;
- **(iv) Negative of negative:** $-(-x) = x$

We will only prove **(i)** and **(iv)**.

Proof. First, we prove **(i)**. Suppose $x, y, z \in F$ such that $x + y = x + z$. Then, as $-x \in F$, we have

$$\begin{aligned} ((-x)) + x + y &= ((-x)) + x + z \\ ((-x) + x) + y &= ((-x) + x) + z \quad \text{by associativity of } + \\ 0 + y &= 0 + z \quad \text{since 0 is the additive identity in } F \end{aligned}$$

and we conclude that $y = z$. $\square$

We then prove **(iv)**.

Proof. Recall that $x + (-x) = 0$. The trick now is to consider

$$-(-x) + (-x) = 0 \quad \text{which again follows by the axiom for negation!}$$

As such,

$$\begin{aligned} x + (-x) &= -(-x) + (-x) \\ x &= -(-x) \quad \text{by the cancellation property in } \textbf{(i)} \end{aligned}$$

so **(iv)** holds. $\square$

Proposition 1.2. The axioms for multiplication in Definition 1.1 imply the following statements: for all $x, y, z \in F$,

- **(i) Cancellation for $\cdot$:** if $x \neq 0$ and $xy = xz$, then $y = z$;
- **(ii) Uniqueness of multiplicative identity:** if $x \neq 0$ and $xy = x$, then $y = 1$;
- **(iii) Uniqueness of reciprocal:** if $x \neq 0$ and $xy = 1$, then $y = 1/x$;
- **(iv) Reciprocal of reciprocal:** if $x \neq 0$, then $1/(1/x) = x$

Proposition 1.3. The field axioms (Definition 1.1) imply the following statements: for all $x, y \in F$,

- **(i)** $0x = 0$;
- **(ii)** if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$
- **(iii)** $(-x)y = -(xy) = x(-y)$
- **(iv)** $(-x)(-y) = xy$

We now discuss what it means for a set to be ordered.

Definition 1.2 (partial order)**.** Let S be a set. A partial ordering relation on S is a relation $\leq$ on S satisfying the following properties:

- **(i) Reflexivity:** for all $x \in S$, we have $x \leq x$
- **(ii) Transitivity:** for all $x, y, z \in S$, we have $x \leq y$ and $y \leq z$ imply $x \leq z$
- **(iii) Antisymmetry:** for all $x, y \in S$, we have $x \leq y$ and $y \leq x$ implies $x = y$

Definition 1.3 (total order)**.** A total ordering relation on S is partial ordering relation $\leq$ (Definition 1.2) on S which also satisies the following property that $\leq$ is comparable:

$$\text{for all } x, y \in S \quad \text{we have} \quad x \leq y \text{ or } y \leq x.$$

Example 1.4. Let S be a set. Then, the subset relation $\subseteq$ on $\mathcal{P}(S)$ is a partial ordering but not a total ordering when $|S| > 1$.

Definition 1.4 (ordered field)**.** An ordered field consists of a field F and a total ordering $\leq$ on F saitsyfing the following properties:

- **(i) $\leq$ is compatible with $+$:** for all $x, y, z \in F$, we have

$$x \leq y \quad \text{implies} \quad x + z \leq y + z$$

- **(ii) $\leq$ is compatible with $\cdot$:** for all $x, y, z \in F$, we have

$$x \leq y \text{ and } z > 0 \quad \text{implies} \quad xz \leq yz$$

Definition 1.5. If

$$x > 0 \text{ we call } x \text{ positive} \quad \text{and if} \quad x < 0 \text{ we call } x \text{ negative} \quad \text{and if}$$
$$x \geq 0 \text{ we call } x \text{ non-negative} \quad \text{and if} \quad x \leq 0 \text{ we call } x \text{ non-positive}$$

Definition 1.6. We have

$$\begin{aligned} F_{>0} &= \{x \in F : x > 0\} \\ F_{<0} &= \{x \in F : x < 0\} \\ F_{\geq 0} &= \{x \in F : x \geq 0\} = F_{>0} \cup \{0\} \\ F_{\leq 0} &= \{x \in F : x \leq 0\} = F_{<0} \cup \{0\} \end{aligned}$$

Example 1.5. $\mathbb{Q}$ given with the usual ordering $\leq$ is an ordered field. We will eventually construct $\mathbb{R}$ as an ordered field.

Proposition 1.4. Let F be an ordered field. Then,

$$\text{for all } x, y \in F \quad \text{we have} \quad x \leq y \text{ if and only if } -x \geq -y.$$

In particular, $F_{<0} = -F_{>0}$ and $F_{\leq 0} = -F_{\geq 0}$.

Proof. We first prove the forward direction. If $x \leq y$, we take $z = (-x) + (-y)$ in F. As such, $x + z \leq y + z$, which implies $-y \leq -x$. For the reverse direction, we apply the same idea to $(x, y) = (-y, -x)$ to obtain $-(-x) \leq -(-y)$. As such, $x \leq y$. □

Proposition 1.5 (closure properties and trichotomy). For any ordered field F,

(i) $F_{>0}$ **is closed under addition:** $F_{>0} + F_{>0} \subseteq F_{>0}$

(ii) $F_{>0}$ **is closed under multiplication:** $F_{>0} \cdot F_{>0} \subseteq F_{>0}$

(iii) Trichotomy: $F = F_{>0} \sqcup \{0\} \cup (-F_{>0})$

Proposition 1.6. For any ordered field F, the following hold:

(i) for all $x \in F$, we have $x^2 \geq 0$

(ii) for all $x, y \in F$ such that $0 < x < y$, we have $0 < 1/y < 1/x$

Proof. We first prove **(i)**. Suppose $x \geq 0$. Then, $x^2 = x \cdot x \geq 0 \cdot x = 0$ by the compatibility of $\leq$ with $\cdot$ (recall **(ii)** of Definition 1.4). If $x < 0$, then $-x > 0$, so $x^2 = (-x)(-x) > 0 \cdot (-x) = 0$ again by **(ii)** of Definition 1.4.

We then prove **(ii)**. Suppose $x > 0$. If $x^{-1} \leq 0$, then $0 = x \cdot 0 \geq x \cdot x^{-1} = 1$, which is a contradiction. As such, we must have $x^{-1} > 0$. If $0 < x < y$, then $xy > 0$ since $F_{>0}$ is closed under multiplication ((**ii**) of Proposition 1.5). As such, $(xy)^{-1} > 0$. Hence,

$$0 < y^{-1} = x \cdot (xy)^{-1} < y \cdot (xy)^{-1} = x^{-1}$$

by the compatibility of $\leq$ with $\cdot$ as mentioned in **(ii)** of Definition 1.4. □

Proposition 1.7 (field characteristic). Let F be an ordered field. Then,

$$\text{for all } n \in \mathbb{N} \quad \text{we have} \quad n \cdot 1 = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ terms}} \quad \text{in } F.$$

In Abstract Algebra, we say that ordered fields have characteristic zero.

Proof. We shall induct on n. The base case $n = 1$ is trivial as $1 > 0$ in F. Next, for any $n \in \mathbb{N}$, if $n \cdot 1 > 0$ in F, then $(n+1) \cdot 1 = n \cdot 1 + 1 > 0$ because $n \cdot 1 > 0$ by the inductive hypothesis and $1 > 0$ trivially. As such, the proposition holds. $\square$

For those who are interested in Abstract Algebra, Definition 1.7 would appeal to you.

Definition 1.7. Let F be an ordered field. Then, F is of characteristic zero. Also, there exists a unique *homomorphism* of fields

$$\iota : \mathbb{Q} \hookrightarrow F \quad \text{called} \quad \text{the canonical inclusion of } \mathbb{Q} \text{ into } F.$$

Moreover, ι is injective and order-preserving.

Via the canonical inclusion $\iota : \mathbb{Q} \hookrightarrow F$ of $\mathbb{Q}$ into F, we will identify $\mathbb{Q}$ with $\iota(\mathbb{Q}) \subseteq F$ and regard $\mathbb{Q}$ as a subfield of F. All these will be covered in MA3201.

Remark 1.2. It follows that ordered fields must be infinite. Also, ordered fields cannot be algebraically closed. To see why, we note that $x^2 + 1 = 0$ has no solution in the ordered field F.

1.2 Supremum, Infimum and Completeness

Definition 1.8 (upper and lower bound). Let S be an ordered set, i.e. a set given with a total ordering. We say that a subset $E \subseteq S$ is

$$\begin{array}{rlll} \text{bounded above} & \text{if and only if} & \text{there exists } B \in S \text{ such that for all } x \in E & \text{we have } x \leq B \\ \text{bounded below} & \text{if and only if} & \text{there exists } A \in S \text{ such that for all } x \in E & \text{we have } A \leq x \\ \text{bounded} & \text{if and only if} & \text{it is bounded above and bounded below} & \end{array}$$

We say that

$$\begin{array}{ll} A \in S & \text{is a lower bound of } E \text{ in } S \\ B \in S & \text{is an upper bound of } E \text{ in } S \end{array}$$

Definition 1.9 (supremum and infimum). Let S be an ordered set and $E \subseteq S$ be any subset. A real number α is the supremum (least upper bound or LUB) of E if

$$\alpha \text{ is an upper bound of } E \quad \text{and} \quad \alpha \leq u \text{ for every upper bound } u \in E, \text{ i.e. } \alpha = \sup(E).$$

A real number β is the infimum (greatest lower bound or GLB) of E if

$$\beta \text{ is a lower bound of } E \quad \text{and} \quad \beta \geq u \text{ for every lower bound } u \in E, \text{ i.e. } \beta = \inf(E).$$

Proposition 1.8. For an ordered set S, let $E \subseteq S$. Then, the set of upper bounds of E in S is always a subset of S. However, it may be empty. We remark that

$$\text{the set of upper bounds} = \emptyset \quad \text{if and only if} \quad E \text{ is not bounded above in } S.$$

Example 1.6. Take $S = \mathbb{Q}$ and $E = \mathbb{Z}$. Then, $E \subseteq S$, and we note that the set of upper bounds of $\mathbb{Z}$ in $\mathbb{Q}$ is $\emptyset$ as the $\sup(\mathbb{Z})$ does not exist.

Remark 1.3. The supremum and infimum of a set may or may not be elements of the set.

Example 1.7. Consider

$$E = \{x \in \mathbb{R} : 0 < x < 1\} \quad \text{where } \inf(E) = 0 \notin E \text{ and } \sup(E) = 1 \notin E.$$

Lemma 1.1 (supremum is unique)**.** Let S be an ordered set. Given $E \subseteq S$,

$$\text{if there exists a least upper bound of } E \text{ in } S \quad \text{then} \quad \sup(E) \text{ is unique.}$$

As mentioned, we write $\sup(E) \in S$ for the unique least upper bound of E in S if it exists.

Proof. The proof is very straightforward. Suppose both α and α' are least upper bounds of E in S. Then, one can show that $\alpha \leq \alpha'$ and $\alpha' \leq \alpha$ by using the two conditions mentioned in Definition 1.9. □

At this juncture, we note that a number of properties of the infimum, or greatest lower bound of a set, have not been discussed. These draw parallelisms with the definition of the supremum (both in Definition 1.9).

Definition 1.10 (least upper bound property)**.** An ordered set S has the least upper bound property if and only if for any non-empty subset $E \subseteq S$ which is bounded above, there exists a least upper bound $\sup(E) \in S$ of E in S.

Example 1.8. Let S_0 be an ordered set†, and let $S \subseteq S_0$ be any finite subset. Then, S, which is regarded as an ordered set, has the least upper bound property (Definition 1.10). In fact, for any non-empty subset $E \subseteq S$ (which is necessarily finite since any subset of a finite set is also finite),

$$\sup(E) = \max(E) \quad \text{exists in } S \text{ (in fact in } E\text{)}.$$

Lemma 1.2. $\mathbb{Z}$, as an ordered set, has the least upper bound property.

Proof. Suppose $E \subseteq \mathbb{Z}$ is any non-empty subset which is bounded above by $b_0 \in \mathbb{Z}$. Then, the set

$$b_0 \setminus E = \{b_0 - x \in \mathbb{Z} : x \in E\} = \{k \in \mathbb{Z} : b_0 - k \in E\} \quad \text{is a non-empty subset of } \mathbb{Z}_{\geq 0}.$$

Note that $b_0 \setminus E$ is indeed non-empty as $E \neq \emptyset$. By the well-ordering property of $\mathbb{Z}_{\geq 0}$, there exists a smallest element $k_0 \in b_0 \setminus E$. As such,

$$\sup(E) = \max(E) = b_0 - k_0 \quad \text{exists in } S.$$

□

We then continue our discussion by showing that $\mathbb{Q}$ does not have the least upper bound property. There are some things to address first. One would know that the equation

$$p^2 = 2 \quad \text{is not satisfied by any } p \in \mathbb{Q}.$$

†If you are unable to appreciate this example well, always make reference to sets, or number systems, that you already know which would be applicable here. For example, we can take $S_0 = \mathbb{Q}$. Consequently as we would see later, $S \subseteq S_0$ is a finite subset of the rationals. Suppose $S = \{-1/2, 3, 10/7\}$ and $E = \{-1/2, 10/7\}$. Then, $\sup(E)$ exists and it is equal to $\max(E) = 10/7$.

This shows that $\sqrt{2}$ is irrational, and consequently, $\mathbb{Q}$ does not have the least upper bound property. Anyway, the proof using the unique factorisation of $\mathbb{Z}$ is as follows:

$$p = \frac{a}{b} \quad \text{for some } a, b \in \mathbb{Z} \text{ and } b \neq 0.$$

Then, consider the prime factorisations of a and b to obtain

$$p = \frac{p_1^{\alpha_1} \dots p_r^{\alpha_r}}{q_1^{\beta_1} \dots q_s^{\beta_s}} \quad \text{so} \quad p_1^{2\alpha_1} \dots p_r^{\alpha_r} = 2 \cdot q_1^{\beta_1} \dots q_s^{\beta_s}.$$

The exponent of 2 on the LHS is even but it is odd on the RHS, resulting in a contradiction.

Now, let

$$A = \left\{ p \in \mathbb{Q}^+ : p^2 < 2 \right\}.$$

Note that A is non-empty and bounded above in $\mathbb{Q}$ since $1 \in A$ and for all $p \in A$, we have $p > 0$ and $p^2 < 2$, so we must have $p < 2$. We shall prove that A contains no largest number. More explicitly,

$$\text{for every } p \in A \quad \text{there exists } q \in A \text{ such that } p < q.$$

Now, for every $p \in A$, we construct q as follows:

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2} \in \mathbb{Q}^+.$$

Also,

$$q^2 - 2 = \left(\frac{2p + 2}{p + 2} \right)^2 - 2 = \frac{2\left(p^2 - 2\right)}{(p + 2)^2}.$$

Since $p \in A$, then $p^2 - 2 < 0$, so $q > p$ and $q^2 - 2 < 0$. Hence, $q \in A$ and is $> p$. As such, A contains no largest number. Anyway, here is a geometrical interpretation of the relationship between p and q (Figure 1). By constructing the line segment joining $\left(p, p^2 - 2\right)$ and $(2, 2)$ and defining $(q, 0)$ to be the point where this line intersects the x-axis, one can indeed deduce that

$$q = p - \frac{p^2 - 2}{p + 2}.$$

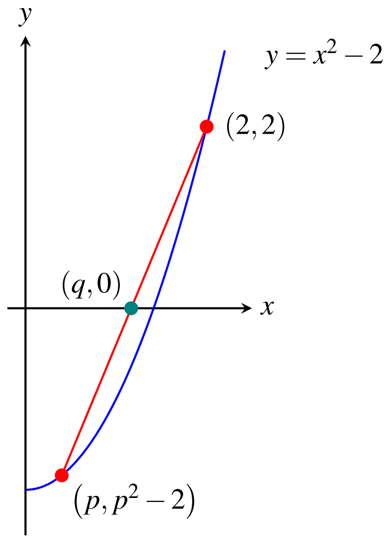


Figure 1: Graph of $y = x^2 - 2$

We are very close to showing that $\mathbb{Q}$ does not have the least upper bound property. More explicitly, for every p in the set of upper bounds of A in $\mathbb{Q}$, one can deduce that there exists q in this set such that $q < p$. As such, this set will not contain a smallest element.

Previously, we showed that A contains no largest number, so no element of A can be an upper bound of A. Similarly, for every p in the set of upper bounds of A in $\mathbb{Q}$, we construct q as follows (Figure 2):

$$q = p - \frac{p^2 - 2}{2p} = \frac{p^2 + 2}{2p} \in \mathbb{Q}^+$$

Also, as $p^2 - 2 > 0$, it follows that $q < p$, so

$$q^2 - 2 = \left(\frac{p^2 + 2}{2p}\right)^2 - 2 = \left(\frac{p^2 - 2}{2p}\right)^2 > 0$$

so $q \notin A$. As such, q is in the set of upper bounds of A in $\mathbb{Q}$.

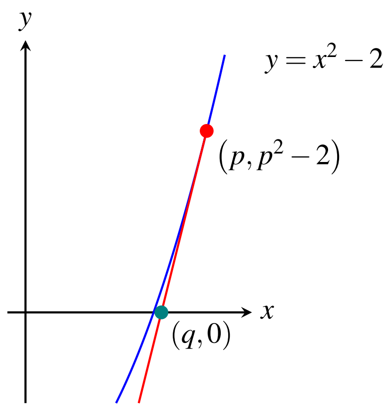


Figure 2: Graph of $y = x^2 - 2$

It follows that $\mathbb{Q}$ does not have the least upper bound property.

Proposition 1.9. Let S be an ordered set with the least upper bound property. Then, it also has the greatest lower bound property. That is to say, for any non-empty subset $B \subseteq S$ which is bounded below,

$$\text{there exists} \quad \text{a greatest lower bound } \inf(B) \in S \text{ of } B \text{ in } S.$$

Proof. Suppose $B \neq \emptyset$ is bounded below. Then, the set of lower bounds of B in S is non-empty and L is bounded above. By the least upper bound property of S, $\alpha = \sup(L)$ exists in S.

We claim that $\alpha = \inf(B)$ as well. We first prove that α is a lower bound of B. Note that for all $x \in B$, x is also in the set of upper bounds of L in S so $\alpha \leq x$. As such, α is also in the set of upper bounds of L in S. Next, we justify that α is the greatest among all lower bounds of B, which holds because $\alpha = \sup(L)$. $\square$

Example 1.9 (Bartle and Sherbert p. 31 Question 12)**.** Let a,b,c,d be numbers satisfying $0 < a < b$ and $c < d < 0$. Give an example where $ac < bd$, and one where $bd < ac$.

Solution. For the first part, we can choose $a = 1$, $b = 2$, $c = -3$, $d = -1$ so that

$$ac = -3 \text{ and } bd = -2 \quad \text{so} \quad ac < bd.$$

For the second part, we can choose $a = 1$, $b = 2$, $c = -2$, $d = -2$ so that

$$bd = -4 \text{ and } ac = -2 \quad \text{so} \quad bd < ac.$$

$\square$

Example 1.10 (Bartle and Sherbert p. 31 Question 14)**.** If $0 \leq a < b$, show that $a^2 \leq ab < b^2$. Show by example that it does *not* follow that $a^2 < ab < b^2$.

Solution. Suppose we are given that $0 \leq a < b$. We first prove that $a^2 \leq ab$, which is equivalent to showing that $ab - a^2 \geq 0$. As such, $a(b-a) \geq 0$. Since $a \geq 0$ and $b > a$ implies $b - a > 0$, then it follows that their product is non-positive, i.e. $a(b-a) \geq 0$.

We then prove that $ab < b^2$, which is equivalent to showing that $b^2 - ab > 0$. As such, $b(b-a) > 0$. Since $b > 0$ and $b > a$ implies $b - a > 0$, their product is positive, i.e. $b(b-a) > 0$.

Having said all these, we show by example that

$$0 \leq a < b \quad \text{does not imply} \quad a^2 < ab < b^2.$$

We choose $a = 0$ so $a^2 = 0$ and $ab = 0$, so the inequality $a^2 < ab$ does not hold. $\square$

Example 1.11 (Bartle and Sherbert p. 31 Question 17)**.** Show the following: If $a \in \mathbb{R}$ is such that

$$0 \leq a \leq \varepsilon \text{ for every } \varepsilon > 0 \quad \text{then} \quad a = 0.$$

Solution. Since $\varepsilon > 0$ is arbitrary, we can choose $\varepsilon = a/2$, so $a \leq a/2$. As such, $a/2 \leq 0$, which implies $a \leq 0$. Combining with the fact that $a \geq 0$, we conclude that $a = 0$. $\square$

Example 1.12 (Bartle and Sherbert p. 31 Question 18)**.** Let $a,b \in \mathbb{R}$, and suppose that for every $\varepsilon > 0$, we have $a \leq b + \varepsilon$. Show that $a \leq b$.

Solution. Suppose on the contrary that $a > b$. Then, $a - b > 0$. Choose $\varepsilon = (a-b)/2 > 0$, so

$$b+\varepsilon-a = b+\frac{a-b}{2}-a = \frac{2b+a-b-2a}{2} = \frac{b-a}{2} < 0.$$

This implies $b+\varepsilon > a$ but this contradicts the fact that $a \leq b+\varepsilon$. To conclude, we must have $a \leq b$. □

Example 1.13 (Bartle and Sherbert p. 39 Question 1)**.** Let

$$S_1 = \{x \in \mathbb{R} : x \geq 0\}.$$

Show in detail that the set S_1 has lower bounds but no upper bounds. Show that $\inf(S_1) = 0$.

Solution. We claim that the set

$$A = \{y \in \mathbb{R} : y \leq 0\} \quad \text{is} \quad \text{the set of lower bounds of } S_1.$$

Let $x \in S_1$ be an arbitrary element. Then, $x \geq 0$. Moreover, for any $y \in \mathbb{R}_{\leq 0}$, we have $x \geq 0 \geq y$, which implies that S_1 has lower bounds and they are all contained in A.

Next, we prove that S has no upper bound. Suppose on the contrary that it has one, say M. Then, $M \in \mathbb{R}_{\geq 0}$ is such that for every $x \in S_1$, we have $x \leq M$. Then, consider the inequality $x \leq M < M+1$. As such, $M+1$ is an upper bound of S_1. By definition of an upper bound, we must have $M+1 \leq M$, which leads to a contradiction. We conclude that S_1 has no upper bound.

Lastly, we prove that $\inf(S_1) = 0$. Recall that A is the set of lower bounds of S_1. As the greatest value of A is 0, then by definition of infimum (greatest lower bound), we conclude that $\inf(S_1) = 0$. □

Example 1.14 (Bartle and Sherbert p. 39 Question 2)**.** Let

$$S_2 = \{x \in \mathbb{R} : x > 0\}.$$

Does S_2 have lower bounds? Does S_2 have upper bounds? Does $\inf(S_2)$ exist? Does $\sup(S_2)$ exist? Prove your statements.

Solution. Similar to Example 1.13, one can show that S_2 has lower bounds (take for example 0) but does not have any upper bound. Next, we claim that $\inf(S_2) = 0$. Consider the set

$$B = \{y \in \mathbb{R} : y \leq 0\} \quad \text{which is} \quad \text{the set of lower bounds of } S_2.$$

One can use the argument in Example 1.13 to justify this. Then, the greatest element of B is 0, so $\inf(S_2) = 0$.

Lastly, we claim that $\sup(S_2)$ does not exist. This follows from the fact that S_2 is not bounded above, so S_2 does not have a least upper bound. □

Example 1.15 (Bartle and Sherbert p. 39 Question 3)**.** Let

$$S_3 = \{1/n : n \in \mathbb{N}\}.$$

Show that $\sup(S_3) = 1$ and $\inf(S_3) \geq 0$.

Solution. Let $\sup(S_3) = \alpha$. Then, for all $x \in S_3$, we must have $x \le \alpha$. That is to say, for any $n \in \mathbb{N}$, we must have $1/n \le \alpha$. We note that the sequence

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \text{ is strictly decreasing} \quad \text{as} \quad \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0.$$

As $\mathbb{N}$ satisfies the well-ordering property, it has a least element, which is 1. So, $1/1 = 1$ is the largest value of S_3, i.e. 1 is an upper bound for S_3. We then prove that 1 is indeed the least upper bound. Suppose on the contrary that there exists $\varepsilon > 0$ such that

$$1 - \varepsilon \quad \text{is} \quad \text{the least upper bound for } S_3 \quad \text{where} \quad n \in \mathbb{N}.$$

We claim that there exists $m \in \mathbb{N}$ such that

$$1 - \varepsilon < \frac{1}{m} \quad \text{or equivalently} \quad \varepsilon > 1 - \frac{1}{m} > 0.$$

This leads to a contradiction.

Next, we prove that $\inf(S_3) \ge 0$. Suppose $\inf(S_3) = \beta$. Then, as $1/n > 0$ for all $n \in \mathbb{N}$, then $\beta \ge 0$. In fact, we can further show that $\beta = 0$. Suppose there exists another lower bound $\beta' \ge 0$. Then, there exists $N \in \mathbb{N}$ such that $1/N < \beta'$, contradicting the fact that β' is a lower bound. We conclude that $\inf(S_3) = 0$. □

Example 1.16 (Bartle and Sherbert p. 39 Question 4)**.** Let

$$S_4 = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$$

Find $\inf(S_4)$ and $\sup(S_4)$.

Solution. For any $x \in S_4$, we have

$$x = \begin{cases} 1 + 1/n & \text{if } n \text{ is odd;} \\ 1 - 1/n & \text{if } n \text{ is even.} \end{cases}$$

Clearly, $\inf(S_4) = 1/2$ and $\sup(S_4) = 2$. □

Example 1.17 (Bartle and Sherbert p. 40 Question 5)**.** Find the infimum and supremum, if they exist, of each of the following sets:

(a) $A = \{x \in \mathbb{R} : 2x + 5 > 0\}$
(b) $B = \{x \in \mathbb{R} : x + 2 \ge x^2\}$
(c) $C = \{x \in \mathbb{R} : x < 1/x\}$
(d) $D = \{x \in \mathbb{R} : x^2 - 2x - 5 < 0\}$

Solution.

(a) The inequality is equivalent to $x > -5/2$, so $\sup(A)$ does not exist but $\inf(A) = -5/2$.
(b) The solution to the inequality is $-1 \le x \le 2$, so $\sup(B) = 2$ and $\inf(B) = -1$[†].
(c) We have

$$\frac{x^2 - 1}{x} < 0 \quad \text{so} \quad \frac{(x+1)(x-1)}{x} < 0.$$

Hence, $x \in (-\infty, -1) \cup (0, 1)$, so $\sup(C) = 1$ and $\inf(C) = -1$.

[†]Actually, to really argue this, we note that the solution set to the inequality is a compact set (just to jump the gun here, we can apply what is known as the Heine-Borel theorem. It states that for a subset of the Euclidean n-space $S \subseteq \mathbb{R}^n$, S is compact if and only if S is closed and bounded). In a compact set, we have $\sup(S) = \max(S)$ and $\inf(S) = \min(S)$.

(d) The solution to the inequality is $1-\sqrt{6}<x<1+\sqrt{6}$ so $\sup(D)=1+\sqrt{6}$ and $\inf(D)=1-\sqrt{6}$. $\square$

Example 1.18 (Bartle and Sherbert p. 44 Question 1)**.** Show that

$$\sup\left\{1-\frac{1}{n}:n\in\mathbb{N}\right\}=1.$$

Solution. Let S be the mentioned set and suppose $\sup(S)=\alpha$. Then,

$$\text{for all } n\in\mathbb{N} \quad \text{we have} \quad 1-\frac{1}{n}\leq\alpha.$$

Since the sequence $\{1-1/n\}_{n=1}^{\infty}$ is decreasing and bounded above by 1, then S is bounded above by 1. Proving that α is indeed 1 is trivial (we discussed this method multiple times). $\square$

Let us try to better understand the least upper bound property.

Example 1.19. Let $E\subseteq\mathbb{R}$ be any non-empty subset that is bounded above. For any $a\in\mathbb{R}$, consider the set

$$a+E=\{a+x\in\mathbb{R}:x\in E\} \quad \text{which is also non-empty and bounded above.}$$

Then, we have

$$\sup(a+E)=a+\sup(E) \quad \text{in } \mathbb{R}.$$

We will justify this result, i.e. show the equality of two real numbers. One common way to go about proving this is to show that the LHS $\leq$ RHS and RHS $\leq$ LHS directly. However, we see that proving the latter directly is difficult, so we will resort to using contradiction.

Proof. We first prove that $\sup(a+E)=a+\sup(E)$. Note that for any $y\in a+E$, there exists $x\in E$ such that $y=a+x$. So, $x\leq\sup(E)$. Adding a to both sides of the inequality yields $y\leq a+$E, so $a+\sup(E)$ is an upper bound of $a+E$. As such, $\sup(a+E)\leq a+\sup(E)$.

We then prove that LHS $<$ RHS leads to a contradiction, which would assert that RHS $\leq$ LHS. Suppose

$$\sup(a+E)<a+\sup(E) \quad \text{or equivalently} \quad \sup(E)>\sup(a+E)-a.$$

We claim that $\sup(a+E)-a$ is still an upper bound for E. To see why, for all $x\in E$, we have $a+x\in a+E$ so $a+x\leq a+\sup(a+E)$. As such, $x\leq\sup(a+E)-a$, contradicting the least upper bound property of $\sup(E)$. $\square$

Lemma 1.3. Let S be an ordered set, and suppose $E\subseteq S$. Let u be an upper bound of E. Then,

$$u=\sup(E) \quad \text{if and only if} \quad \text{for all } \varepsilon>0 \text{ there exists } x\in E \text{ such that } u-\varepsilon<x.$$

We refer to Figure 3 for an illustration of Lemma 1.3.

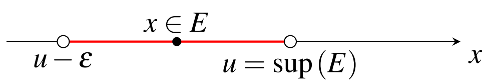


Figure 3: Illustration of the supremum condition in Lemma 1.3

Example 1.20 (Bartle and Sherbert p. 40 Question 14)**.** Let S be a set that is bounded below. Prove that a lower bound w of S is the infimum of S if and only if for any $\varepsilon>0$, there exists $t\in S$ such that $t<w+\varepsilon$.

Solution. For the forward direction, suppose $w = \inf(S)$. So, for any $t \in S$, we have $w \le t$. Suppose on the contrary that

$$t \ge w + \varepsilon \quad \text{for every } t \in S.$$

So, $w + \varepsilon$ is also a lower bound for S. However, by definition of the infimum, w is the greatest lower bound for S, which implies $w \ge w + \varepsilon$. As such, $\varepsilon \le 0$, which is a contradiction.

For the reverse direction, suppose for any $\varepsilon > 0$, there exists $t \in S$ such that $t < w + \varepsilon$. We already know that w is a lower bound for S. Suppose v is another lower bound for w. We claim that $v \le w$. Suppose on the contrary that $v > w$. Choose $\varepsilon = w - v > 0$. Then, we have

$$t < w + \varepsilon \quad \text{which implies} \quad t < w + (v - w) = v.$$

As such, v cannot be a lower bound for S, which leads to a contradiction. □

Axiom 1.1. Every non-empty subset of $\mathbb{R}$ which is

$$\begin{matrix} \text{bounded above} \\ \text{bounded below} \end{matrix} \text{ has a } \begin{matrix} \text{supremum;} \\ \text{infimum.} \end{matrix}$$

Example 1.21 (Bartle and Sherbert p. 36 Question 5)**.** If $a < x < b$ and $a < y < b$, show that $|x - y| < b - a$.

Solution. Suppose on the contrary that $|x - y| \ge b - a$. Then, either

$$x - y \ge b - a \quad \text{or} \quad x - y \le a - b.$$

Note that $a < y < b$ implies $-b < -y < -a$, so $x - y < b - a$, contradicting the claim that $x - y \ge b - a$. Similarly, $b - a < x - y$, but again this leads to a contradiction. Hence, we must have $|x - y| < b - a$. □

Example 1.22 (MA2108 AY19/20 Sem 1 Tutorial 1)**.** Let $a, b \in \mathbb{R}$. Show that

$$\max(a,b) = \frac{1}{2}(a + b + |a - b|) \quad \text{and} \quad \min(a,b) = \frac{1}{2}(a + b - |a - b|).$$

Solution. We consider two cases, namely $a \ge b$ and $a < b$. If $a \ge b$, then $a - b \ge 0$, then

$$\frac{1}{2}(a + b + |a - b|) = \frac{1}{2}(a + b + a - b) = a = \max(a,b).$$

Similarly,

$$\frac{1}{2}(a + b - |a - b| = \frac{1}{2}(a + b - (a - b) = b = \min(a,b).$$

The case where $a < b$ has similar working. □

Example 1.23 (Bartle and Sherbert p. 40 Question 7)**.** If a set $S \subseteq \mathbb{R}$ contains one of its upper bounds, show that this upper bound is the supremum of S.

Solution. Let $u \in \mathbb{S}$ be an upper bound for S. Suppose v is another upper bound for S such that $v < u$. Choosing $S \ni s = u$, there exists $s \in S$ such that $v < s$, which contradicts our claim that $v < u$. As such, we must have $u \le v$, i.e. if we have another upper bound v of S, then $u \le v$. We conclude that $u = \sup(S)$. □

Example 1.24 (Bartle and Sherbert p. 40 Question 8)**.** Let $S \subseteq \mathbb{R}$ be nonempty. Show that

$$u \in \mathbb{R} \text{ is an upper bound of } S \quad \text{if and only if} \quad \text{the conditions } t \in \mathbb{R} \text{ and } t > u \text{ imply } t \notin S.$$

Solution. We first prove the forward direction. Suppose $u \in \mathbb{R}$ is an upper bound of S. Then, for all $s \in S$, we have $s \leq u$. Say $t \in \mathbb{R}$ is such that $t > u$. Suppose on the contrary that $t \in S$. Then, because S is bounded above by u, we must have $t \leq u$, contradicting $t > u$. We conclude that $t \notin S$.

For the reverse direction, we argue by contradiction — say u is not an upper bound of S. Then, there exists $s_0 \in S$ such that $u < s_0$. Let $t = s_0$, then $t > u$, but this contradicts our hypothesis as any $t \in \mathbb{R}$ such that $t > u$ implies $t \notin S$. However, we have $t = s_0 \in S$, which is a contradiction. $\square$

Proposition 1.10 (Archimedean property). For any $x, y \in \mathbb{R}$ such that $0 < x < y$,

$$\text{there exists } n \in \mathbb{N} \quad \text{such that} \quad nx > y \text{ in } \mathbb{R}.$$

Corollary 1.1. For any $\varepsilon \in \mathbb{R}_{>0}$,

$$\text{there exists } n \in \mathbb{N} \quad \text{such that} \quad n\varepsilon > 1 \text{ in } \mathbb{R}.$$

Theorem 1.1 (density theorem). The rational numbers are dense in $\mathbb{R}$, i.e.

$$\text{if } a, b \in \mathbb{R} \text{ such that } a < b \quad \text{then} \quad \text{there exists } r \in \mathbb{Q} \text{ such that } a < r < b.$$

In short, we are always able to find another rational number that lies between two real numbers.

Corollary 1.2. The irrationals are dense in $\mathbb{R}$, i.e.

$$\text{if } a, b \in \mathbb{R} \text{ such that } a < b \quad \text{then} \quad \text{there exists } x \in \mathbb{Q}' \text{ such that } a < x < b.$$

Example 1.25. $\mathbb{Q}$ satisfies the Archimedean property. Also, $\mathbb{Q}$ is dense in $\mathbb{Q}$ with respect to ordering — clearly, for any rational numbers a and b such that $a < b$, we can always find another rational number r strictly in between them. Take for example, $r = \frac{1}{2}(a+b)$.

Theorem 1.2 (existence and uniqueness of radicals). Let $x > 0$ and $n \in \mathbb{N}$. Then, exists a unique positive real number y such that $y^n = x$. The number y is known as the positive n^{th} root of x and thus,

$$y = \sqrt[n]{x} = x^{1/n}.$$

Proof. The uniqueness claim is quite obvious — suppose we have two positive real numbers $0 < y_1 < y_2$. Then, $0 < y_1^n < y_2^n$. As such, given that $y_1^n = y_2^n$, it implies $y_1 = y_2$.

We then prove the existence claim[†]. Let E denote the set consisting of all positive real numbers t such that $t^n < x$, i.e.

$$E = \{t \in \mathbb{R} : t > 0 \text{ and } t^n < x\}.$$

First, we claim that $y = \sup E$ exists in $\mathbb{R}$. By the least upper bound property of $\mathbb{R}$, it suffices to show that $E \neq \emptyset$ and E is bounded above. Consider $t = x/(1+x) \in \mathbb{R}$[‡]. Since $x > 0$, then we get $0 \leq t < 1$ and $t < x$. By

[†]As mentioned by Prof. Chin Chee Whye, when you encounter the proof in Rudin's book for the first time (referring to 'Principles of Mathematical Analysis'), you will feel so angry to the extent that you will throw the book away.

[‡]Actually, it is fairly intuitive to consider this function (though we will jump the gun). We can think of the problem as follows: construct a sequence of positive numbers x_n that is increasing, bounded between 0 and 1. Try to think of $x_n = 1 - 1/n$. However, as the sequence must be defined at index 0, we simply do a translation to obtain $x_n = 1 - 1/(n+1)$. One checks that $x_n = n/(n+1)$.

induction on k, we note that for all $k \in \mathbb{N}$, we have $0 \le t^k \le t < 1$, so $t^n \le t < x$. To see why, note that $t^k \ge 0$ is clear since $t \ge 0$. As such, it suffices to prove that $t^k \le t$. Equivalently, $t\left(t^{k-1}-1\right) \le 0$, so $t^{k-1}-1 \le 0$, so $t^{k-1} \le 1$ (in fact, this inequality is strict) which holds by the induction hypothesis.

The above shows that $t \in E$, so $E \neq \emptyset$.

We then claim that $1+x \in \mathbb{R}$ is an upper bound of E, i.e. for all $t \in E$, one has $t \le 1+x$. Suppose on the contrary that there exists $t \in E$ such that $t > 1+x$, i.e. $t > 1$ and $t > x$. Again, by induction on k, we note that for all $k \in \mathbb{N}$, we have $t^k \ge t+1$. To see why, we consider

$$\begin{aligned} t^{k+1} &= t \cdot t^k \\ &\ge t(t+1) \quad \text{by induction hypothesis} \\ &> t+1 \quad \text{since } t > 1 \end{aligned}$$

Hence, $t^n \ge t > x$, contradicting the hypothesis that $t \in E$. So, $t \notin E$.

Lastly, we shall prove that $y = \sup E$, i.e. y is a positive real number satisfying $y^n = x$. This is clear for the case when $n = 1$. As such, we will prove the claim for $n \ge 2$. To do this, we will show that $y^n < x$ and $y^n > x$ both lead to a contradiction. First, note that for any $a, b \in \mathbb{R}$ such that $0 < a < b$, we have the inequality

$$b^n - a^n < (b-a)\, nb^{n-1}.$$

To see why this inequality holds, recall the geometric series formula

$$\left(\frac{b}{a}\right)^n - 1 = \left[1 + \frac{b}{a} + \left(\frac{b}{a}\right)^2 + \ldots + \left(\frac{b}{a}\right)^{n-1}\right]\left(\frac{b}{a} - 1\right)$$

so

$$\frac{b^n - a^n}{a^n} = \frac{1}{a^{n-1}}\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1}\right)\left(\frac{b-a}{a}\right).$$

Since the expressions on each side contain $1/a^n$, it follows that

$$\begin{aligned} b^n - a^n &= (b-a)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1}\right) \\ &< (b-a)\left(b^{n-1} + b^{n-2} \cdot b + b^{n-3} \cdot b^2 + \ldots + b^{n-1}\right) \quad \text{since } a < b \\ &= (b-a)\, nb^{n-1} \end{aligned}$$

Assume that $y^n < x$. Choose h such that

$$0 < h < \min\left\{1, \frac{x - y^n}{n(y+1)^{n-1}}\right\}.$$

Setting $a = y$ and $b = y + h$, we have

$$\begin{aligned} b^n - a^n &= (y+h)^n - y^n \\ &< hn(y+h)^{n-1} \quad \text{since } b^n - a^n < (b-a)\, nb^{n-1} \text{ as deduced earlier} \\ &\le hn(y+1)^{n-1} \quad \text{since } h \le 1 \\ &= x - y^n \end{aligned}$$

So, $(y+h)^n < x$. Also, $y+h \in E$. Since $y+h > y$, this contradicts the fact that y is an upper bound of E.

Next, assume that $y^n > x$. Again, we will show that this leads to a contradiction. Choose

$$k \quad \text{such that} \quad 0 < k < \frac{y^n - x}{ny^{n-1}}.$$

Then, as $y^n > x$ implies $y^n - x > 0$, and $ny^{n-1} > 0$, it follows that $k > 0$. Next, by setting $b = y$ and $a = y - k$, we have

$$\begin{aligned} b^n - a^n &= y^n - (y-k)^n \\ &< kny^{n-1} \quad \text{since } b^n - a^n < (b-a)\, nb^{n-1} \text{ as deduced earlier} \end{aligned}$$

Since $y^n - (y-k)^n < y^n - x$ is equivalent to saying that $(y-k)^n > x$, our goal is to choose $k > 0$ such that $kny^{n-1} < y^n - x$. To be precise, the chosen value of k should be

$$k = \frac{y^n - x}{ny^{n-1}}.$$

If $t \geq y - k$, then

$$y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x.$$

As such, $t^n > x$ and $t \notin E$. It follows that $y - x$ is an upper bound of E. However, $y - k < y$, contradicting the fact that y is the least upper bound of E. To conclude, we must have $y^n = x$. □

We analyse the proof of Theorem 1.2. Recall that

$$y = \sup(E) \quad \text{where} \quad E = \{t \in \mathbb{R} : t > 0 \text{ and } t^n < x\}.$$

If $y^n < x$, then as shown in Figure 4, we define $\varepsilon = x - y^n$ in $\mathbb{R}_{>0}$. Try to choose a number of the form $y^n + \delta$, where $\delta \in \mathbb{R}_{>0}$ and $\delta < \varepsilon$ such that $y^n + \delta$ is of the form $(y+h)^n$, where $h \in \mathbb{R}_{>0}$. Consequently, this would contradict the fact that y is an upper bound for E.

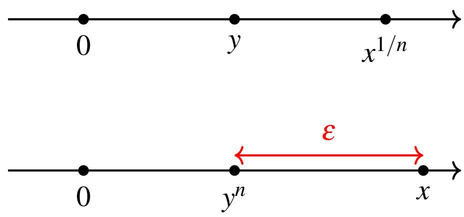


Figure 4

In Figure 5, we wish to choose $h \in \mathbb{R}_{>0}$ such that $0 < (y+h)^n - y^n < \varepsilon$, where $(y+h)^n - y^n = \delta$. In the proof of Theorem 1.2, we used the inequality $b^n - a^n = (b-a)\, nb^{n-1}$ (which follows by considering some finite geometric series). As such, $(y+h)^n - y^n < hn(y+h)^{n-1}$, so it suffices to choose $h \in \mathbb{R}_{>0}$ so that $hn(y+h)^{n-1} < \varepsilon$.

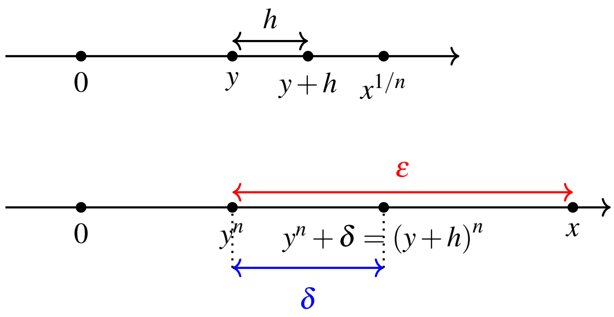


Figure 5

The above is equivalent to choosing $h \in \mathbb{R}_{>0}$ so that

$$h < \frac{\varepsilon}{n\,(y+h)^{n-1}} \quad \text{which is impossible.}$$

On the other hand, if $y^n > x$, then we define $\varepsilon = y^n - x$, which is in $\mathbb{R}_{>0}$ (Figure 6).

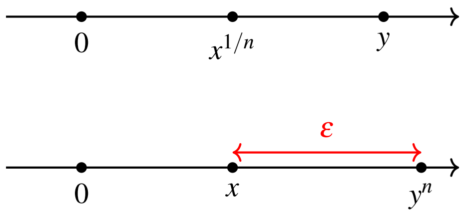


Figure 6

In Figure 7, we wish to choose a real number of the form $y^n - \delta$, where $\delta \in \mathbb{R}_{>0}$ and $\delta \le \varepsilon$ such that $y^n - \delta$ is of the form $(y-k)^n$, where $k \in \mathbb{R}_{>0}$.

Equivalently, we try to choose $k \in \mathbb{R}_{>0}$ such that $0 < y^n - (y-k)^n \le \varepsilon$. By the inequality $b^n - a^n = (b-a)\,nb^{n-1}$, we have $y^n - (y-k)^n < kny^{n-1}$ so it suffices to choose $k \in \mathbb{R}_{>0}$ such that $kny^{n-1} \le \varepsilon$. This suggests to choose

$$k = \frac{\varepsilon}{ny^{n-1}}.$$

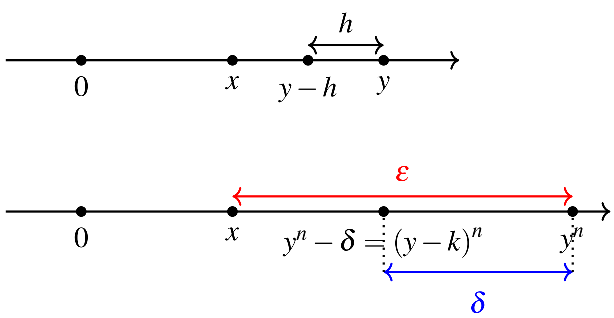


Figure 7

Corollary 1.3. If a and b are positive real numbers and $n \in \mathbb{N}$, then

$$(ab)^{1/n} = a^{1/n} b^{1/n}.$$

Proof. Let $\alpha = a^{1/n}$ and $\beta = b^{1/n}$, so $ab = \alpha^n \beta^n = (\alpha\beta)^n$. By the uniqueness of Theorem 1.2, we are done. □

We have a nice corollary on $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ (Corollary 1.4). This set is defined to be the multiplicative group of real numbers (will encounter in MA2202 and beyond).

Corollary 1.4. Let $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$. Then,

$$\mathbb{R}_{>0} = \left(\mathbb{R}^\times\right)^2 \quad \text{as subsets of } \mathbb{R}.$$

Proof. Let F be an ordered field. The inclusion $F_{>0} \supseteq \left(F^\times\right)^2$ is obvious. In particular, this holds for $F = \mathbb{R}$. For the other inclusion, if $x \in \mathbb{R}_{>0}$, then by Theorem 1.2, there exists $y \in \mathbb{R}_{>0} \subseteq \mathbb{R}^\times$ such that $y^2 = x$. Hence, the result follows. □

Example 1.26 (Bartle and Sherbert p. 40 Question 6)**.** Let S be a non-empty subset of $\mathbb{R}$ that is bounded below. Prove that

$$\inf(S) = -\sup\{-s : s \in S\}.$$

Solution. This problem aims to prove

$$\inf(S) = -\sup(-S).$$

Note that $\inf(S) \leq s$ for all $s \in S$. So, $-\inf(S) \geq -s$, so $-\inf(S)$ is an upper bound for $-S$, which implies

$$-\inf(S) \geq \sup(-S) \quad \text{so} \quad \inf(S) \geq -\sup(-S).$$

For the other direction, since S is bounded below, then $-S$ is bounded above so $-s \leq \sup(-S)$ for all $s \in S$. Hence,

$$s \leq -\sup(-S) \quad \text{which implies} \inf(S) \leq -\sup(-S).$$

Combining both inequalities yields the desired result. □

Example 1.27 (Bartle and Sherbert p. 45 Question 4)**.** Let S be a set of non-negative real numbers that is bounded above. Let $a > 0$, and let $aS = \{as : s \in S\}$. Prove that

$$\sup(aS) = a\sup(S).$$

Solution. Suppose $u = \sup(aS)$. Then, for all $as \in aS$, we have $as \leq u$. Since $a > 0$, we have

$$s \leq \frac{u}{a} = \frac{\sup(aS)}{a} \quad \text{which implies} \quad \sup(S) \leq \frac{\sup(aS)}{a}.$$

We then prove the reverse direction. Suppose $\sup(S) = v$. Then, for all $s \in S$, we have $s \leq v$. As such,

$$as \leq av = a\sup(S) \quad \text{which implies} \quad \sup(aS) \leq a\sup(S).$$

□

Example 1.28 (Bartle and Sherbert p. 45 Question 7)**.** Let A and B be bounded non-empty subsets of $\mathbb{R}$, and let

$$A + B = \{a + b : a \in A, b \in B\}.$$

This is known as the *Minkowski sum* of two sets. Prove that

$$\sup(A+B) = \sup(A) + \sup(B) \quad \text{and} \quad \inf(A+B) = \inf(A) + \inf(B).$$

Solution. We only prove the first result as the second result can be proven similarly. We first prove that

$$\sup(A)+\sup(B)\le\sup(A+B).$$

For all $a\in A$ and $b\in B$, we have

$$a+b\le\sup(A+B).$$

Subtracting b from both sides yields

$$a\le\sup(A+B)-b.$$

If we fix b, we see that $\sup(A+B)-b$ is an upper bound for A. By definition of supremum, we have

$$\sup(A)\le\sup(A+B)-b\quad\text{which implies}\quad b\le\sup(A+B)-\sup(A),$$

i.e. $\sup(A+B)-\sup(A)$ is an upper bound for any b. As such,

$$\sup(B)\le\sup(A+B)-\sup(A)\quad\text{or equivalently}\quad\sup(A)+\sup(B)\le\sup(A+B).$$

We then prove that $\sup(A)+\sup(B)\ge\sup(A+B)$. Since $\sup(A)$ is an upper bound for A, then $a\le\sup(A)$ for all $a\in A$. Similarly, $b\le\sup(B)$ for all $b\in B$. It follows that $a+b\le\sup(A)+\sup(B)$. So, $\sup(A)+\sup(B)\ge\sup(A+B)$. By considering both inequalities, the result follows. □

The real numbers satisfy the *completeness axiom*[†].

Definition 1.11 (completeness of $\mathbb{R}$)**.** There are no gaps or missing points in $\mathbb{R}$.

Corollary 1.5. $\mathbb{N}$ is not bounded above.

Proof. For any $\varepsilon>0$, there exists $n\in\mathbb{N}$ such that $1/n<\varepsilon$. This is justified by setting $x=\varepsilon$ and $y=1$. □

Example 1.29 (Bartle and Sherbert p. 44 Question 2)**.** If

$$S=\left\{\frac{1}{n}-\frac{1}{m}:n,m\in\mathbb{N}\right\},$$

find $\inf(S)$ and $\sup(S)$.

Solution. Note that

$$\frac{1}{n}-\frac{1}{m}<\frac{1}{n}.$$

Since $n_{\min}=1$, then $1=1/1$ is an upper bound for S. To see why, we can fix $n=1$ and then consider the following sequence of numbers:

$$1-\frac{1}{1},1-\frac{1}{2},1-\frac{1}{3},\ldots,1-\frac{1}{m}$$

For large m, the sequence increases and tends to 1, S is bounded above by 1. Next, by the Archimedean property, for any $\varepsilon>0$, there exists $m\in\mathbb{N}$ such that

$$\frac{1}{m}<\varepsilon\quad\text{so}\quad1-\frac{1}{m}>1-\varepsilon.$$

This shows that 1 is an upper bound for S but $1-\varepsilon$ is not an upper bound for S for any $\varepsilon>0$. As such, 1 is the least upper bound for S, so $\sup S=1$.

[†]Definition 1.11 is rather intuitive and simple. In fact, this was coined by Dedekind.

Next, note that

$$\frac{1}{n}-\frac{1}{m}>\frac{1}{n}-1>-1,$$

which implies that -1 is a lower bound for S. Again, to see why, we fix $m=1$ and then consider the following sequence of numbers:

$$\frac{1}{1}-1,\frac{1}{2}-1,\frac{1}{3}-1,\ldots,\frac{1}{n}-1$$

For large n, the sequence decreases and tends to -1, which implies that S is bounded below by -1. Next, by the Archimedean property, for any $\varepsilon>0$, there exists $n\in\mathbb{N}$ such that

$$\frac{1}{n}<\varepsilon \quad \text{so} \quad \frac{1}{n}-1<\varepsilon-1.$$

This shows that -1 is a lower bound for S but $-1+\varepsilon$ is not a lower bound for S for any $\varepsilon>0$. As such, -1 is the greatest lower bound for S, so $\inf S=-1$. ☐

We have a very nice geometric interpretation of Example 1.29. Actually, we can also let

$$S=\left\{\frac{1}{m}-\frac{1}{n}:m,n\in\mathbb{N}\right\} \quad \text{because} \quad m \text{ comes before } n \text{ in the English alphabet.}$$

Consider the following infinite matrix:

$$\begin{array}{cccccc}
\frac{1}{1}-\frac{1}{1} & \frac{1}{1}-\frac{1}{2} & \frac{1}{1}-\frac{1}{3} & \frac{1}{1}-\frac{1}{4} & \cdots & \sup(S)=1 \\
\frac{1}{2}-\frac{1}{1} & \frac{1}{2}-\frac{1}{2} & \frac{1}{2}-\frac{1}{3} & \frac{1}{2}-\frac{1}{4} & \cdots & \vdots \\
\frac{1}{3}-\frac{1}{1} & \frac{1}{3}-\frac{1}{2} & \frac{1}{3}-\frac{1}{3} & \frac{1}{3}-\frac{1}{4} & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\inf(S)=-1 & \cdots & \cdots & \cdots & \cdots & 0
\end{array}$$

The *matrix* is skew-symmetric since its transpose is equal to negative of itself. Next, for any element, as we travel rightwards, its value increases; as we travel downwards, its value decreases. By observation, the *maximum* and *minimum* values of this matrix (technically they should be the supremum and infimum respectively) occur on the boundary[†].

Example 1.30 (Bartle and Sherbert p. 40 Question 9). Let $S\subseteq\mathbb{R}$ be non-empty. Show that if $u=\sup(S)$, then for every number $n\in\mathbb{N}$, the number $u-1/n$ is not an upper bound of S, but the number $u+1/n$ is an upper bound of S.

Solution. Let $u=\sup(S)$ for some $\emptyset\neq S\subseteq\mathbb{R}$. By definition, the supremum u is an upper bound of S, so $u+1/n$ is also an upper bound of S.

We then prove that $u-1/n$ is not an upper bound of S. Suppose on the contrary that it is. Since $u=\sup(S)$, then $u-1/n<u$. As such, there exists $s_0\in S$ such that

$$u-\frac{1}{n}<s_0<u,$$

which is a contradiction as this shows that $u-1/n$ is not an upper bound of S. ☐

Example 1.31 (Bartle and Sherbert p. 44 Question 3). Let $S\subseteq\mathbb{R}$ be non-empty. Prove that if a number u in $\mathbb{R}$ has the properties

[†]There is a nice result in Real Analysis which states that when a function is monotonic on a domain, extrema occur at the boundary. By the term 'monotonic', we mean that the sequence of numbers is either increasing or decreasing. For example, the sequence $1,2,-1,4,\ldots$ is not monotonic but the sequence of positive odd numbers $1,3,5,7,9,\ldots$ is monotonic.

(i) for every $n \in \mathbb{N}$, the number $u - 1/n$ is not an upper bound of S;
(ii) For every $n \in \mathbb{N}$, the number $u + 1/n$ is an upper bound of S,
then $u = \sup(S)$.

Solution. By **(ii)**, for any $s \in S$, we have

$$s \le u + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

Since n can be made arbitrarily large, i.e. $n \to \infty$, then $s \le u$, which holds for all $s \in S$. As such, u is an upper bound of S.

Next, fix some $\varepsilon > 0$. By the Archimedean property, there exists $n_0 \in \mathbb{N}$ such that $1/n_0 \le \varepsilon$. Hence,

$$u - \varepsilon \le u - \frac{1}{n_0}.$$

Since $u - 1/n_0$ is not an upper bound of S by **(i)**, then there exists $s_0 \in S$ such that

$$u - \varepsilon \le u - \frac{1}{n_0} \le s_0.$$

Hence, u is the least upper bound of S, which implies $u = \sup(S)$. □

Example 1.32 (Bartle and Sherbert p. 40 Question 10)**.** Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set. Show that

$$\sup(A \cup B) = \sup\{\sup A, \sup B\}.$$

Solution. Since A and B are bounded subsets of $\mathbb{R}$, then there exist $m_1, m_2, M_1, M_2 \in \mathbb{R}$ such that for any $a \in A$ and $b \in B$, we have

$$m_1 \le a \le M_1 \quad \text{and} \quad m_2 \le b \le M_2.$$

Let $x \in A \cup B$. Then $x \in A$ or $x \in B$, which implies

$$\min\{m_1, m_2\} \le c \le \max\{M_1, M_2\}.$$

This shows that $A \cup B$ is also a bounded subset of $\mathbb{R}$.

Next, we prove that

$$\sup(A \cup B) = \sup\{\sup A, \sup B\}.$$

It suffices to prove that $\sup(A \cup B) = \max\{\sup A, \sup B\}$. From the previous part, we already deduced that $\sup(A \cup B) \le \max\{\sup A, \sup B\}$. To prove the reverse inequality, note that $\sup(A)$ is an upper bound for A, so $\sup(A \cup B) \ge \sup(A)$. A similar argument shows that $\sup(A \cup B) \ge \sup(B)$. Hence,

$$\sup(A \cup B) \ge \max\{\sup A, \sup B\}.$$

The result follows. □

Example 1.33 (Bartle and Sherbert p. 40 Question 11)**.** Let S be a bounded set in $\mathbb{R}$ and let S_0 be a non-empty subset of S. Show that

$$\inf S \le \inf S_0 \le \sup S_0 \le \sup S.$$

Solution. We first prove that $\sup(S_0) \le \sup(S)$. Note that one can deduce that $\inf(S) \le \inf(S_0)$. Suppose on the contrary that $\sup(S_0) > \sup(S)$. Then, $\sup(S)$ is not an upper bound for S_0, which is a contradiction because $S_0 \subseteq S$.

Lastly, we prove that $\inf(S_0) \le \sup(S_0)$. Let $\sup(S_0) = \alpha$ and $\inf(S_0) = \beta$. Let $x \in S_0$. Since β is a lower bound for x, then $x \ge \beta$. Similarly, since α is an upper bound of x, then $x \le \alpha$, which shows that $\beta \le x \le \alpha$. Hence, $\beta \le \alpha$. □

Example 1.34 (Bartle and Sherbert p. 40 Question 12)**.** Let $S \subseteq \mathbb{R}$ and suppose $\sup(S) \in S$. If $u \notin S$, show that

$$\sup\left(S \cup \{u\}\right) = \sup\{\sup(S), u\}.$$

Solution. We consider two cases. Firstly, suppose $u \ge \sup(S)$. Since $\sup(S) \in S$ and is an upper bound for S, then $s \le \sup(S) \le u$ for any $s \in S$. As such, u is an upper bound for $S \cup \{u\}$. If y is another upper bound for $S \cup \{u\}$, then it forces $y \ge u$. Hence, the least upper bound of $S \cup \{u\}$ is u, so $\sup(S \cup \{u\}) = u$. On the other hand, since $\sup = \max$ in this case and $u \ge \sup(S)$, then $\sup\{\sup(S), u\} = u$.

Next, we consider the case when $u < \sup(S)$. Since $\sup(S) \in S$, it is the largest element of S, so for all $s \in S$, we have $s \le \sup(S)$. As such, $\sup(S)$ is an upper bound for $S \cup \{u\}$. Moreover, if y is any upper bound for $S \cup \{u\}$, then $y \ge \sup(S)$, so $\sup(S) = \sup(S \cup \{u\})$. The result follows. □

Let us take a look at Example 1.35. The result states that The result states that for a function of two variables $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ which is bounded above, the order of taking the supremum does not matter.

Example 1.35 (a Fubini-like identity)**.** Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a function which is bounded above. Prove that

$$\sup_{m\in\mathbb{N}} \sup_{n\in\mathbb{N}} f(m,n) = \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} f(m,n) = \sup_{(m,n)\in\mathbb{N}\times\mathbb{N}} f(m,n).$$

Here is a nice geometric interpretation of the problem. We can view $f(m,n)$ as a surface. The supremum represents the highest *peak* or maximum value attained by this surface.

The process of taking $\sup_{m\in\mathbb{N}}$ first means that for each fixed n, we look at the highest value along the column $\{(m,n)\}_{m\in\mathbb{N}}$. Then, we take the supremum of these column-wise maxima over all n, which corresponds to finding the highest peak among these values. Conversely, taking $\sup_{n\in\mathbb{N}}$ first means scanning along the *row* $\{(m,n)\}_{n\in\mathbb{N}}$ for each fixed m, then finding the highest peak among those row-wise maxima.

On the other hand, directly taking the supremum over all ordered pairs $(m,n) \in \mathbb{N} \times \mathbb{N}$ means looking at all points at once and finding the highest value. Since taking the supremum column-first or row-first still results in scanning all points, they must all yield the same value.

Hence, this result shows that regardless of whether we take the maximum first across rows or columns, we always reach the same overall highest point in the grid. We now formally discuss the solution.

Solution. Since f is bounded above, we have

$$f(m,n) \le \sup_{m\in\mathbb{N}} f(m,n).$$

Taking the supremum over all $(m,n) \in \mathbb{N} \times \mathbb{N}$, we have

$$\sup_{(m,n)\in\mathbb{N}\times\mathbb{N}} f(m,n) \le \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} f(m,n).$$

Conversely, we have

$$\sup_{n\in\mathbb{N}}\sup_{m\in\mathbb{N}} f(m,n) \le \sup_{(m,n)\in\mathbb{N}\times\mathbb{N}} f(m,n) \quad \text{so it follows that} \quad \sup_{n\in\mathbb{N}}\sup_{m\in\mathbb{N}} f(m,n) = \sup_{(m,n)\in\mathbb{N}\times\mathbb{N}} f(m,n).$$

In a similar fashion, one can deduce that

$$\sup_{m\in\mathbb{N}}\sup_{n\in\mathbb{N}} f(m,n) = \sup_{(m,n)\in\mathbb{N}\times\mathbb{N}} f(m,n)$$

The result follows. □

Theorem 1.3. If n is non-square, then $\sqrt{n}$ is irrational.

Proof. Suppose on the contrary that $\sqrt{n}$ is rational, where n is non-square. Then,

$$\sqrt{n} = p/q \text{ implies } nq^2 = p^2 \quad \text{where } p,q\in\mathbb{N}, q\neq 0 \text{ and } \gcd(p,q)=1.$$

We consider the prime factorisations of p^2 and q^2, each one of them having an even number of primes. Thus, n must also have an even number of primes. As n is non-square, there exists at least a prime with an odd multiplicity, which is a contradiction. □

Theorem 1.4. Every non-empty interval $I\subseteq\mathbb{R}$ contains infinitely many rational numbers and infinitely many irrational numbers.

1.3 Important Inequalities

Bernoulli's inequality (named after Jacob Bernoulli) is an inequality that approximates exponentiations of $1+x$. We discuss a widely-used version of this result.

Theorem 1.5 (Bernoulli's inequality). For every $r\in\mathbb{Z}_{\ge 0}$ and $x\ge -1$, we have

$$(1+x)^n \ge 1+nx.$$

The inequality is strict if $x\neq 0$ and $r\ge 2$.

One can use induction to prove Theorem 1.5.

Example 1.36 (MA2108 AY19/20 Sem 1 Tutorial 1). Use Bernoulli's inequality to deduce that for any integer $n>1$, the following hold:

$$\left(1-\frac{1}{n^2}\right)^n > 1-\frac{1}{n} \quad \text{and} \quad \left(1+\frac{1}{n-1}\right)^{n-1} < \left(1+\frac{1}{n}\right)^n$$

Solution. The first result is obvious by setting $x=-1/n^2$ in Theorem 1.5. For the second result, we wish to prove that

$$\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n-1}\right)^{n-1}} > 1.$$

Using some algebraic manipulation, we have

$$1+\frac{1}{n-1} = \frac{n}{n-1} = \frac{1}{1-\frac{1}{n}}.$$

Hence,

$$\text{LHS} = \left(1+\frac{1}{n}\right)^n \left(1-\frac{1}{n}\right)^{n-1} = \left(1+\frac{1}{n}\right)^n \left(1-\frac{1}{n}\right)^n \left(1-\frac{1}{n}\right)^{-1} = \left(1-\frac{1}{n^2}\right)^n \left(1-\frac{1}{n}\right)^{-1}$$

which is >1. Here, the inequality follows from the first result. □

Theorem 1.6 (QM-AM-GM-HM inequality). Let $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$. Let

$$\begin{aligned} Q(n) &= \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2} && \text{denote} && \text{the quadratic mean} \\ A(n) &= \frac{1}{n}\sum_{i=1}^{n} x_i && \text{denote} && \text{the arithmetic mean} \\ G(n) &= \sqrt[n]{\prod_{i=1}^{n} x_i} && \text{denote} && \text{the geometric mean} \\ H(n) &= n\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1} && \text{denote} && \text{the harmonic mean} \end{aligned}$$

Then, $Q(n) \geq A(n) \geq G(n) \geq H(n)$. Equality is attained if and only if $x_1 = \ldots = x_n$.

Remark 1.4. The quadratic mean $Q(n)$ is also referred to as root mean square or RMS.

We first prove that $Q(n) \geq A(n)$.

Proof. By the Cauchy-Schwarz inequality,

$$n\sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i\right)^2 \quad \text{which implies} \quad \frac{n[Q(n)]^2}{n} \geq [nA(n)]^2.$$

With some simple rearrangement, the result follows. □

Example 1.37 (MA2108S AY16/17 Sem 2 Homework 5). For each $n \in \mathbb{Z}^+$, let

$$a_n = \left(1+\frac{1}{n}\right)^n \quad \text{and} \quad b_n = \left(1+\frac{1}{n}\right)^{n+1}.$$

(a) Show that a_n is strictly monotonically increasing.

(b) Show that b_n is strictly monotonically decreasing.

Hint: Use the GM-HM Inequality.

(c) Show that for each $n \in \mathbb{Z}^+$, one has $a_n < b_n$.

Solution.

(a) A special form of the AM-GM inequality states that

$$\frac{x+ny}{n+1} \geq (xy^n)^{1/(n+1)}.$$

Setting $x = 1$ and $y = 1 + 1/n$, we have

$$1+\frac{1}{n+1} > \left(1+\frac{1}{n}\right)^{n/(n+1)} \quad \text{so} \quad \left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$$

which shows that $a_{n+1} > a_n$. Note that the inequality is strict since $x \neq y$.

(b) Similar to **(a)**.

(c) Let $(1+1/n) = u$. Then, $b_n - a_n = u^{n+1} - u^n = u^n/n$, which is positive. □

In fact, both sequences a_n and b_n in Example 1.37 converge in $\mathbb{R}$ and to the same limit. Justifying this requires the use of the monotone convergence theorem (Theorem 2.9) which will be covered in due course. The real number which is the common limit of the sequences a_n and b_n is called Euler's number[†] and it is denoted by e.

We now return to the proof of the QM-AM-GM-HM inequality (Theorem 1.6). There are numerous proofs of the AM-GM inequality like using backward-forward induction (Cauchy), considering e^x (Pólya), Lagrange Multipliers (MA2104) etc. This proof hinges on Jensen's inequality.

Theorem 1.7 (Jensen's inequality)**.** For a concave function $f(x)$,

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \le f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

Proof. Consider the logarithmic function $f(x) = \ln x$, where $x \in \mathbb{R}^+$. It can be easily verified that $f(x)$ is concave as $f''(x) = -1/x^2 < 0$ (this is a simple exercise using knowledge from MA2002). We wish to prove

$$\ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \ln\left(\sqrt[n]{\prod_{i=1}^{n} x_i}\right).$$

Using Jensen's inequality (Theorem 1.7),

$$\frac{1}{n}\sum_{i=1}^{n} \ln(x_i) \le \ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

Note that

$$\sum_{i=1}^{n} \ln(x_i) = \ln(x_1) + \ln(x_2) + \ldots + \ln(x_n) = \ln\left(\prod_{i=1}^{n} x_i\right).$$

As such, the inequality becomes

$$\ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \frac{1}{n}\ln\left(\prod_{i=1}^{n} x_i\right).$$

With some simple rearrangement, the AM-GM inequality follows. □

Lastly, we will prove the GM-HM Inequality using the AM-GM Inequality.

Proof. Note that

$$\prod_{i=1}^{n} \frac{1}{x_i} = \left(\prod_{i=1}^{n} x_i\right)^{-1},$$

so we have

$$\frac{n/H(n)}{n} \ge \frac{1}{G(n)}.$$

Upon rearranging, we are done. □

Theorem 1.8 (triangle inequality)**.** For $x, y \in \mathbb{R}$,

$$|x+y| \le |x| + |y|$$

and equality is attained if and only if $xy \ge 0$.

[†] Not to be confused with Euler's constant as this typically denotes the Euler-Mascheroni constant γ.

Corollary 1.6. The following hold:

(i) $|x-y| \leq |x|+|y|$

(ii) **Reverse triangle inequality:** $||x|-|y|| \leq |x-y|$

Proof. **(i)** can be easily proven by replacing $-y$ with y. We now prove **(ii)**. Write x as $x-y+y$ and y as $y-x+x$. Hence,

$$\begin{aligned} |x| &= |x-y+y| \leq |x-y|+|y| \\ |y| &= |y-x+x| \leq |y-x|+|x| = |x-y|+|x| \end{aligned}$$

As such, $|x|-|y| \leq |x-y|$ and $|x|-|y| \leq -|y-x|$, and taking the absolute value of $|x|-|y|$, the result follows. □

Example 1.38 (Bartle and Sherbert p. 35 Question 2)**.** If $a,b \in \mathbb{R}$, show that

$$|a+b| = |a|+|b| \quad \text{if and only if} \quad ab \geq 0.$$

Solution. We first prove the forward direction. Suppose $|a+b| = |a|+|b|$. Squaring both sides yields

$$|a+b|^2 = |a|^2 + 2|ab| + |b|^2.$$

We note that $|a|^2 = a^2$ for any $a \in \mathbb{R}$. As such,

$$\begin{aligned} (a+b)^2 &= a^2 + 2|ab| + b^2 \\ a^2 + 2ab + b^2 &= a^2 + 2|ab| + b^2 \end{aligned}$$

which implies $ab = |ab|$. Hence, $ab \geq 0$.

Conversely, suppose we know that $ab \geq 0$. Then, either

$$a \geq 0 \text{ and } b \geq 0 \quad \text{or} \quad a \leq 0 \text{ and } b \leq 0.$$

For the first case, $a+b$ is the sum of two non-negative numbers, which is also non-negative. Hence, $|a+b| = a+b$. Since $|a| = a$ and $|b| = b$, it follows that $|a+b| = |a|+|b|$. For the second case, $a+b$ is the sum of two non-positive numbers, which is also non-positive. As such, $|a+b| = -(a+b)$. Similarly, we also know that $|a| = -a$ and $|b| = -b$. The result follows. □

Corollary 1.7 (generalised triangle inequality)**.** For $x_1, \ldots, x_n \in \mathbb{R}$,

$$\left| \sum_{i=1}^{n} x_i \right| \leq \sum_{i=1}^{n} |x_i|.$$

Proof. Repeatedly apply the triangle inequality (Theorem 1.8). □

Example 1.39 (MA2108 AY19/20 Sem 1 Tutorial 1)**.** Prove that if $x,y \in \mathbb{R}$, $y \neq 0$ and $|x| \leq \frac{|y|}{2}$, then

$$\frac{|x|}{|x-y|} \leq 1.$$

Solution. We wish to prove that $|x| \leq |x-y|$. Using the given inequality, we apply the triangle inequality, so

$$|x| \leq |y|/2 = \frac{|y+x-x|}{2} \leq \frac{|y-x|+|x|}{2}.$$

The result follows with some simple rearrangement and using the property that $|x-y| = |y-x|$. □

Example 1.40 (MA2108S AY16/17 Sem 2 Homework 5; Chebyshev's sum inequality)**.** Let $n \in \mathbb{N}$. Show that for any elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathbb{R}$ with $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$, one has Chebyshev's inequality, i.e.

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} a_i b_i.$$

Solution. For any $1 \leq i, j \leq n$, we have

$$\begin{aligned} (a_i - a_j)(b_i - b_j) &\geq 0 \\ a_i b_i + a_j b_j &\geq a_i b_j + a_j b_i \end{aligned}$$

Taking the double sum over all i and j on both sides,

$$\begin{aligned} \sum_{j=1}^{n}\sum_{i=1}^{n} a_i b_i + a_j b_j &\geq \sum_{j=1}^{n}\sum_{i=1}^{n} a_i b_j + a_j b_i \\ \sum_{j=1}^{n}\sum_{i=1}^{n} a_i b_i + \sum_{j=1}^{n}\sum_{i=1}^{n} a_j b_j &\geq \sum_{j=1}^{n}\sum_{i=1}^{n} a_i b_j + \sum_{j=1}^{n}\sum_{i=1}^{n} a_j b_i \\ n\sum_{i=1}^{n} a_i b_i + n\sum_{j=1}^{n} a_j b_j &\geq \sum_{i=1}^{n} a_i \sum_{j=1}^{n} b_j + \sum_{i=1}^{n} b_i \sum_{j=1}^{n} a_j \\ 2n\sum_{i=1}^{n} a_i b_i &\geq 2\left(\sum_{i=1}^{n} a_i \sum_{j=1}^{n} b_j\right) \\ \sum_{i=1}^{n} a_i b_i &\geq \frac{1}{n}\sum_{i=1}^{n} a_i \sum_{j=1}^{n} b_j \end{aligned}$$

Changing the right sum of b_j's to run from $i = 1$ to $i = n$, and dividing both sides by n, the result follows. □

Example 1.41 (MA2108S AY16/17 Sem 2 Homework 5; Hölder's inequality)**.** Let $n \in \mathbb{N}$. Show that for any $a_1, \ldots, a_n$ in $\mathbb{R}$ with $a_i \geq 0$ for each $1 \leq i \leq n$ and for any $p \in \mathbb{N}$, one has the inequality

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^p \leq \frac{1}{n}\sum_{i=1}^{n} a_i^p.$$

Solution. We use Hölder's inequality, which states that for $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathbb{R}^+$ and $p, q > 1$ such that $1/p + 1/q = 1$,

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p}\left(\sum_{i=1}^{n} b_i^q\right)^{1/q}$$

Set $q = p/(p-1)$ so

$$\begin{aligned} \sum_{i=1}^{n} a_i b_i &\leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p}\left(\sum_{i=1}^{n} b_i^{p/(p-1)}\right)^{(p-1)/p} \\ \left(\sum_{i=1}^{n} a_i b_i\right)^p &\leq \left(\sum_{i=1}^{n} a_i^p\right)\left(\sum_{i=1}^{n} b_i^{p/(p-1)}\right)^{p-1} \end{aligned}$$

We can set $b_i = 1$ for all $1 \leq i \leq n$ so the inequality becomes

$$\left(\sum_{i=1}^{n} a_i\right)^p \leq \left(\sum_{i=1}^{n} a_i^p\right) n^{p-1}$$

and with some simple algebraic manipulation, the result follows. □

Chapter 2
Sequences

2.1
Limit of a Sequence

Definition 2.1 (sequence)**.** Let X be a set. A sequence in X is

$$\text{a function } x \text{ with domain } \mathbb{N} \quad \text{i.e. } x : \mathbb{N} \to X$$

which assigns to each natural number n an element $x_n \in X$. The notation x_n is commonly used to denote the image of n under x, meaning $x_n = x(n)$.

Some authors might also use $X(n)$ in Definition 2.1 but x_n is the more standard notation.

We give some examples of sequences.

Example 2.1 (constant sequence)**.** Given $p \in X$, for all $n \in \mathbb{N}$, define $x_n = p$ so we obtain the constant sequence of value p in X.

Example 2.2. We have the exponential sequence 2^n and the factorial sequence $n!$.

Example 2.3 (recursively defined sequences)**.** Also known as recurrence relations, we can apply the recursion theorem for $\mathbb{N}$ (*formally*) to construct a map $\mathbb{N} \to X$. For example, we have the Fibonacci sequence

$$\text{for all } n \in \mathbb{N} \text{ we have } x_{n+1} = x_n + x_{n-1} \quad \text{defined by the initial conditions} \quad x_0 = x_1 = 1.$$

Definition 2.2 (absolute value)**.** Let F be an ordered field. The absolute value on F is the map

$$|\cdot| : F \to F_{\geq 0} \quad \text{defined by} \quad x \mapsto |x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Proposition 2.1. For any $x, y \in F$, we have the following:

(i) Positive-definiteness: $|x| \geq 0$ in F and equality holds if and only if $x = 0$ in F

(ii) Multiplicativity: $|xy| = |x||y|$ in $\mathbb{F}_{\geq 0}$

(iii) Triangle inequality: $|x+y| \leq |x| + |y|$ (recall Theorem 1.8)

Definition 2.3 (neighbourhood)**.** Let F be an ordered field. For any $a \in F$ and $\varepsilon > 0$, define

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\} = (a - \varepsilon, a + \varepsilon) \text{ (Figure 8)} \quad \text{to be} \quad \text{the } \varepsilon\text{-neighbourhood of } a.$$

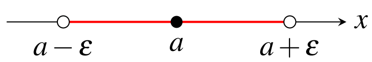


Figure 8: ε-neighbourhood of a

Definition 2.4 (formal definition of limit). Let F be an ordered field and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in F. We say that L is the limit of the sequence if

$$\text{for every } \varepsilon > 0 \text{ there exists } K \in \mathbb{N} \quad \text{such that} \quad \text{for all } n \geq K \text{ we have } |x_n - L| < \varepsilon.$$

Equivalently, $x_n \in V_\varepsilon(L)$. If L exists, then we say that $\{x_n\}_{n\in\mathbb{N}}$ converges to L in F (or simply $\{x_n\}_{n\in\mathbb{N}}$ is convergent); $\{x_n\}_{n\in\mathbb{N}}$ diverges otherwise.

Theorem 2.1 (uniqueness of limit of sequence). The limit of a sequence $\{x_n\}_{n\in\mathbb{N}}$, if it exists, is unique. That is to say, if $L, L' \in F$ for some ordered field F such that

$$\lim_{n\to\infty} x_n = L \text{ and } \lim_{n\to\infty} x_n = L' \quad \text{then} \quad L = L'.$$

Proof. Suppose on the contrary that L and L' are two distinct limits of $\{x_n\}_{n\in\mathbb{N}}$. By way of contradiction, say $L \neq L'$. Then, we can write $\varepsilon = |L - L'| \in F_{>0}$. The trick is to observe that $\varepsilon/2 \in F_{>0}$. Since $x_n \to L$, there exists $K_1 \in \mathbb{N}$ such that for all $n \geq K_1$, the inequality

$$|x_n - L| < \varepsilon' = \frac{\varepsilon}{2} \quad \text{holds.}$$

Similarly, as $x_n \to L'$, then there exists $K_2 \in \mathbb{N}$ such that for all $n \geq K_2$, the inequality

$$|x_n - L'| < \varepsilon' = \frac{\varepsilon}{2} \quad \text{holds.}$$

We define $K = \max\{K_1, K_2\}$, which is also $\in \mathbb{N}$. Then, for all $n \geq K$, we have

$$|L - L'| = |L - x_n + x_n - L'| \leq |x_n - L| + |x_n - L'| < 2\varepsilon' = \varepsilon.$$

Here, the first inequality follows from the triangle inequality. Since ε is arbitrary, we can set $|L - L'| = 0$, resulting in $L = L'$, contradicting the earlier assumption that L and L' are distinct. □

Remark 2.1. The triangle inequality is a helpful tool when finding limits. Note that changing a finite number of terms in a sequence does not affect its convergence or its limit.

Same as the formal definition of a limit in MA2002, to prove that a given sequence x_n converges to L, we first express $|x_n - L|$ in terms of n, and find a *simple* upper bound, L, for it. Then, let $\varepsilon > 0$ be arbitrary. We find $K \in \mathbb{N}$ such that

$$\text{for all } n \geq K \text{ we have } L < \varepsilon \quad \text{or equivalently} \quad |x_n - L| < \varepsilon.$$

Example 2.4. Prove that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Solution. Let $\varepsilon > 0$. By the Archimedean property (Proposition 1.10), there exists $K \in \mathbb{N}$ such that $K > 1/\varepsilon$. So, if $n \geq K$, then $n > 1/\varepsilon$. As such, $1/n < \varepsilon$. We conclude that for all $n \geq K$, $|1/n - 0| < \varepsilon$. □

Example 2.5. Prove that

$$\lim_{n\to\infty} \frac{2n^2+1}{n^2+3n} = 2.$$

Solution. We have

$$\left|\frac{2n^2+1}{n^2+3n} - 2\right| = \left|\frac{1-6n}{n^2+3n}\right| \leq \frac{1+6n}{n^2+3n} < \frac{1+6n}{n^2} < \frac{n+6n}{n^2} = \frac{7}{n}.$$

Let $\varepsilon > 0$ be given. Choose $K \in \mathbb{N}$ such that $K > 7/\varepsilon$. Then, for all $n \geq K$, we have

$$\left|\frac{2n^2+1}{n^2+3n} - 2\right| < \frac{7}{n} \leq \frac{7}{K} < \varepsilon$$

and the result follows. □

Example 2.6 (Bartle and Sherbert p. 62 Question 5)**.** Show that

(a) $\lim\limits_{n\to\infty} \frac{n}{n^2+1} = 0$ **(b)** $\lim\limits_{n\to\infty} \frac{2n}{n+1} = 2$ **(c)** $\lim\limits_{n\to\infty} \frac{3n+1}{2n+5} = \frac{3}{2}$ **(d)** $\lim\limits_{n\to\infty} \frac{n^2-1}{2n^2+3} = \frac{1}{2}$

Solution.

(a) Let $\varepsilon > 0$ be arbitrary. Choose $N = 1/\lceil \varepsilon \rceil$ in $\mathbb{N}$. Then, for all $n \geq N$, we have

$$\left|\frac{n}{n^2+1} - 0\right| = \left|\frac{n}{n^2+1}\right| \leq \left|\frac{n}{n^2}\right| = \frac{1}{|n|} \leq \frac{1}{N} < \varepsilon.$$

(b) Let $\varepsilon > 0$ be arbitrary. Choose $N = 2/\lceil \varepsilon \rceil$ in $\mathbb{N}$. Then, for all $n \geq N$, we have

$$\left|\frac{2n}{n+1} - 2\right| = \left|\frac{2n-2n-2}{n+1}\right| = \frac{2}{|n|} \leq \frac{2}{N} < \varepsilon.$$

(c) Let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 13/4\varepsilon \rceil$. Then, for all $n \geq N$, we have

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|\frac{13}{2(2n+5)}\right| \leq \frac{13}{4|n|} \leq \frac{13}{4N} < \varepsilon.$$

(d) Let $\varepsilon > 0$ be arbitrary. Choose $N = \left\lceil \sqrt{5/4\varepsilon} \right\rceil$. Then, for all $n \geq N$, we have

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \frac{5}{2|2n^2+3|} \leq \frac{5}{4n^2} \leq \frac{5}{4N^2} < \varepsilon.$$

□

Example 2.7 (Bartle and Sherbert p. 62 Question 6)**.** Show that

(a) $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n+7}} = 0$ **(b)** $\lim\limits_{n\to\infty} \frac{2n}{n+2} = 2$ **(c)** $\lim\limits_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$ **(d)** $\lim\limits_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0$

Solution.

(a) Let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 1/\varepsilon^2 \rceil$. Then, for all $n \geq N$, we have

$$\left|\frac{1}{\sqrt{n+7}} - 0\right| \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \varepsilon.$$

(b) Let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 4/\varepsilon \rceil$. Then, for all $n \geq N$, we have

$$\left|\frac{2n}{n+2} - 2\right| = \left|\frac{4}{n+2}\right| \leq \frac{4}{|n|} \leq \frac{4}{N} < \varepsilon.$$

(c) Let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 1/\varepsilon^2 \rceil$. Then, for all $n \geq N$, we have

$$\left|\frac{\sqrt{n}}{n+1} - 0\right| = \frac{\sqrt{n}}{n+1} \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \varepsilon.$$

(d) Let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 1/\sqrt{\varepsilon} \rceil$. Then, for all $n \geq N$, we have

$$\left|\frac{(-1)^n}{n^2+1} - 0\right| = \frac{1}{n^2+1} \leq \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon.$$

□

Example 2.8 (Bartle and Sherbert p. 69 Question 1)**.** For x_n given by the following formulas, establish either the convergence or the divergence of the sequence $X = \{x_n\}_{n\in\mathbb{N}}$:

(a) $x_n = \dfrac{n}{n+1}$ **(b)** $x_n = \dfrac{(-1)^n n}{n+1}$ **(c)** $x_n = \dfrac{n^2}{n+1}$ **(d)** $x_n = \dfrac{2n^2+3}{n^2+1}$

Solution.

(a) The sequence converges to 1. We will formally prove this. Let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 1/\varepsilon \rceil$ in $\mathbb{N}$. Then, for all $n \geq N$, we have

$$\left| \frac{n}{n+1} - 1 \right| = \left| \frac{1}{n+1} \right| \leq \frac{1}{|n|} \leq \frac{1}{N} < \varepsilon.$$

(b) We claim that the sequence diverges. Suppose on the contrary that the limit is L. Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left| \frac{(-1)^n n}{n+1} - L \right| < \varepsilon.$$

Let $\varepsilon = 1$. If n is even, then there exists $k \in \mathbb{Z}$ such that $n = 2k$ so

$$\left| \frac{(-1)^{2k} \cdot 2k}{2k+1} - L \right| < 1 \quad \text{so} \quad \left| \frac{2k}{2k+1} - L \right| < 1.$$

Upon expansion, we have

$$L - 1 < \frac{2k}{2k+1} < L + 1 \quad \text{and we see that} \quad \lim_{k \to \infty} \frac{2k}{2k+1} = 1.$$

On the other hand, if n is odd, then there exists $k \in \mathbb{Z}$ such that $n = 2k+1$ so

$$\left| \frac{(-1)^{2k+1} \cdot (2k+1)}{2k+2} - L \right| < 1 \quad \text{so} \quad \left| \frac{-2k-1}{2k+2} - L \right| < 1.$$

Upon expansion, we have

$$L - 1 < \frac{-2k-1}{2k+2} < L + 1 \quad \text{but however} \quad \lim_{k \to \infty} \frac{-2k-1}{2k+2} = -1.$$

Since both limits are different, this leads to a contradiction.

(c) We claim that the sequence diverges. To see why, we have the following inequality:

$$\left| \frac{n^2}{n+1} \right| \geq \left| \frac{n^2}{n^2} \right| = 1 \quad \text{so for sufficiently large } n \quad \left| \frac{n^2}{n+1} \right| \geq 1$$

so the sequence diverges.

(d) We claim that the sequence converges to 2. Let $\varepsilon > 0$ be arbitrary. Then, choose $N = \lceil 1/\sqrt{\varepsilon} \rceil$. As such,

$$\left| \frac{2n^2+3}{n^2+1} - 2 \right| = \left| \frac{1}{n^2+1} \right| \leq \left| \frac{1}{n^2} \right| < \frac{1}{N^2} < \varepsilon.$$

□

Example 2.9 (Bartle and Sherbert p. 69 Question 6). Find the limits of the following sequences:

(a) $\displaystyle\lim_{n \to \infty} \left(2 + \frac{1}{n^2} \right)^2$ **(b)** $\displaystyle\lim_{n \to \infty} \frac{(-1)^n}{n+2}$ **(c)** $\displaystyle\lim_{n \to \infty} \frac{\sqrt{n}-1}{\sqrt{n}+1}$ **(d)** $\displaystyle\lim_{n \to \infty} \frac{n+1}{n\sqrt{n}}$

Solution.

(a) 4

(b) We claim that the limit is 0. To see why, let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 1/\varepsilon \rceil$ in $\mathbb{N}$. As such,

$$\left| \frac{(-1)^n}{n+2} - 0 \right| = \frac{1}{|n+2|} \leq \frac{1}{N} < \varepsilon.$$

(c) We have

$$\lim_{n\to\infty} \frac{\sqrt{n}-1}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1-1/\sqrt{n}}{1+1/\sqrt{n}} = 1.$$

(d) The limit is

$$\lim_{n\to\infty} \frac{n+1}{n} \cdot \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 1 \cdot 0 = 0.$$

□

Example 2.10 (Bartle and Sherbert p. 62 Question 12)**.** Show that

$$\lim_{n\to\infty} \left(\sqrt{n^2+1} - n \right) = 0.$$

Solution. We will use the formal definition of a limit to prove that the limit is 0. Before that, to see why one can make this deduction, we have

$$\sqrt{n^2+1} - n = \frac{\left(\sqrt{n^2+1}-n\right)\left(\sqrt{n^2+1}+n\right)}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n}$$

so as $n \to \infty$, the limit goes to zero. We now prove this formally. Let $\varepsilon > 0$ be arbitrary. Then, choose $N = \lceil 1/2\varepsilon \rceil$ in $\mathbb{N}$. So, for all $n \geq N$, we have

$$\begin{aligned}
\left| \sqrt{n^2+1} - n - 0 \right| &= \left| \sqrt{n^2+1} - n \right| \\
&= \left| \frac{\left(\sqrt{n^2+1}-n\right)\left(\sqrt{n^2+1}+n\right)}{\sqrt{n^2+1}+n} \right| \\
&= \left| \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} \right| \\
&= \frac{1}{\sqrt{n^2+1}+n}
\end{aligned}$$

Now, note that $\sqrt{n^2+1} \geq \sqrt{n^2} = n$ so $\sqrt{n^2+1}+n \geq 2n$. Hence,

$$\frac{1}{\sqrt{n^2+1}+n} \leq \frac{1}{2n} \leq \frac{1}{2N} < \varepsilon.$$

□

Example 2.11 (Bartle and Sherbert p. 70 Question 10)**.** Determine the limits of the following sequences:

(a) $\sqrt{4n^2+n} - 2n$

(b) $\sqrt{n^2+5n} - n$

Solution.

(a) We have

$$\lim_{n\to\infty} \left(\sqrt{4n^2+n} - 2n \right) = \lim_{n\to\infty} \frac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n} = \lim_{n\to\infty} \frac{n}{\sqrt{4n^2+n}+2n} = \lim_{n\to\infty} \frac{1}{\sqrt{4+1/n}+2} = \frac{1}{4}.$$

(b) We have

$$\lim_{n\to\infty} \left(\sqrt{n^2+5n} - n \right) = \lim_{n\to\infty} \frac{n^2+5n-n^2}{\sqrt{n^2+5n}+n} = \lim_{n\to\infty} \frac{5}{\sqrt{1+5/n}+1} = \frac{5}{2}.$$

□

Example 2.12 (Bartle and Sherbert p. 70 Question 13). If $a>0, b>0$, show that

$$\lim_{n\to\infty}\left(\sqrt{(n+a)(n+b)}-n\right)=\frac{a+b}{2}.$$

Solution. Let $\varepsilon>0$ be arbitrary. Then, choose $N=\left\lceil\frac{(a-b)^2}{4\varepsilon}\right\rceil$ in $\mathbb{N}$, and also let $k=\frac{1}{2}(a+b)$ for convenience. So, for all $n\geq N$, we have

$$\begin{aligned}
\left|\sqrt{(n+a)(n+b)}-n-\frac{a+b}{2}\right| &= \left|\sqrt{(n+a)(n+b)}-n-k\right| \\
&= \left|\frac{\left(\sqrt{(n+a)(n+b)}-(n+k)\right)\left(\sqrt{(n+a)(n+b)}+(n+k)\right)}{\sqrt{(n+a)(n+b)}+(n+k)}\right| \\
&= \left|\frac{(n+a)(n+b)-(n+k)^2}{\sqrt{(n+a)(n+b)}+(n+k)}\right| \\
&= \left|\frac{n^2+an+bn+ab-n^2-2kn-k^2}{\sqrt{(n+a)(n+b)}+(n+k)}\right| \\
&= \left|\frac{an+bn+ab-(a+b)n-\left(\frac{a+b}{2}\right)^2}{\sqrt{(n+a)(n+b)}+(n+k)}\right|
\end{aligned}$$

At this juncture, note that the numerator simplifies to

$$ab-\left(\frac{a+b}{2}\right)^2=ab-\frac{a^2+2ab+b^2}{4}=-\frac{(a-b)^2}{4}.$$

By considering the denominator, we have

$$\sqrt{(n+a)(n+b)}+(n+k)\geq\sqrt{(n+a)(n+b)}\geq\sqrt{n^2}=n$$

so

$$\left|\sqrt{(n+a)(n+b)}-n-\frac{a+b}{2}\right|\leq\frac{(a-b)^2}{4n}\leq\frac{(a-b)^2}{4N}<\varepsilon.$$

□

In Example 2.12, it was stated that $a,b>0$. This condition is not surprising because the expression $\sqrt{(n+a)(n+b)}$ must be well-defined for all relevant values of n. Specifically, the square root function requires that its argument be non-negative, meaning that $(n+a)(n+b)>0$.

For this inequality to hold for all sufficiently large n, both $n+a$ and $n+b$ must be either simultaneously non-negative or simultaneously non-positive. If either a or b were negative, there would exist some values of n for which $(n+a)(n+b)<0$, making the square root expression undefined in the real number system. Thus, ensuring that $a,b>0$ guarantees the validity of the expression for all sufficiently large n.

Example 2.13 (Bartle and Sherbert p. 93 Question 3). Show that if $x_n>0$ for all $n\in\mathbb{N}$, then

$$\lim_{n\to\infty}x_n=0\quad\text{if and only if}\quad\lim_{n\to\infty}\frac{1}{x_n}=\infty.$$

Solution. We first prove the forward direction. Suppose

$$\lim_{n\to\infty}x_n=0.$$

Let $\varepsilon > 0$ be arbitrary and set $M = 1/\varepsilon$. Then, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $|x_n| < \varepsilon = 1/M$. Thus, for $n \geq N$, we have $1/x_n > M$, and the result follows.

For the reverse direction, we note that there exists $N \in \mathbb{N}$ such that for $n \geq N$, $1/x_n > M$. Let $\varepsilon > 0$ be arbitrary and set $M = 1/\varepsilon$. Then, $1/x_n > 1/\varepsilon$, so $|x_n| < \varepsilon$. The result follows. □

Example 2.14 (Bartle and Sherbert p. 93 Question 7)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 0.$$

(a) Show that if $\lim x_n = \infty$, then $\lim y_n = \infty$.

(b) Show that if $\{y_n\}_{n\in\mathbb{N}}$ is bounded, then $\lim x_n = 0$.

Solution.

(a) Since

$$\lim_{n\to\infty} x_n = \infty \quad \text{then} \quad \text{there exists } K \in \mathbb{N} \text{ such that for all } n \geq K \text{ we have } x_n > 1.$$

Since

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 0 \quad \text{then} \quad \text{there exists } N \in \mathbb{N} \text{ such that for all } n \geq N \text{ we have } \left|\frac{x_n}{y_n}\right| < \varepsilon.$$

Thus,

$$\frac{1}{|y_n|} < \left|\frac{x_n}{y_n}\right| < \varepsilon \quad \text{for all } n \geq \max\{K, N\},$$

which shows that

$$\lim_{n\to\infty} \frac{1}{y_n} = 0.$$

By Example 2.13, the result follows.

(b) Since y_n is bounded, then there exists $M > 0$ such that $0 < |y_n| \leq M$. We wish to prove that there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, $|x_n| < \varepsilon$. We have $|x_n/y_n| < \varepsilon/M$ so $|x_n| < \varepsilon/M \cdot M = \varepsilon$. □

Definition 2.5 (eventually constant)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an ordered field F. We say that the sequence is eventually constant if and only if there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n = x_N$.

Definition 2.6 (boundedness)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an ordered field F. We say that the sequence is bounded in F if and only if

$$\text{the set } \{x_n \in F : n \in \mathbb{N}\} \quad \text{is} \quad \text{bounded in } F.$$

Theorem 2.2 (limit theorems)**.** Let F be an ordered field and $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be convergent sequences in F. Then, the following properties hold:

(i) $\{x_n\}_{n\in\mathbb{N}}$ is convergent, i.e. if

$$\lim_{n\to\infty} x_n = L \quad \text{then} \quad |x_n| \leq M \text{ for some } M \in \mathbb{R}$$

(ii) **Linearity:** Just like how linear operators (i.e. derivatives and integrals) work, we have a similar result for limits. Suppose $\alpha, \beta \in F$ and

$$\lim_{n\to\infty} x_n = L_1 \quad \text{and} \quad \lim_{n\to\infty} y_n = L_2.$$

Then,

$$\{\alpha x_n \pm \beta y_n\}_{n\in\mathbb{N}} \text{ converges} \quad \text{i.e.} \quad \lim_{n\to\infty} (\alpha x_n \pm \beta y_n) = \alpha L_1 \pm \beta L_2.$$

(iii) Product and quotient: Considering the sequences x_n and y_n as mentioned in **(ii)**,

$$\lim_{n\to\infty} x_n y_n = L_1 L_2 \quad \text{and} \quad \lim_{n\to\infty} \frac{x_n}{y_n} = \frac{L_1}{L_2} \quad \text{provided that } y_n, y \neq 0 \text{ for all } n \in \mathbb{N}$$

(iv) If there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \leq y_n$ in F, then

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n \quad \text{in } F$$

The converse of Theorem 2.2 is not true as not all bounded sequences are convergent.

Example 2.15. As an example, the sequence $x_n = (-1)^n$ is bounded by -1 and 1 and it oscillates about only these two values. We claim that $\{x_n\}_{n\in\mathbb{N}}$ does not converge in F. By way of contradiction, say

$$\lim_{n\to\infty} x_n = L \quad \text{in } F.$$

Set $\varepsilon = 1$. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - L| < 1$. Hence, for odd $n \geq N$, we have $|-1-p| < 1$, which implies $-1 < -1-p < 1$, so $p < 0$. On the other hand, for even $n \geq N$, we have $|1-p| < 1$, which implies $-1 < 1-p < 1$, which implies $p > 0$. This leads to a contradiction!

We first prove **(i)** of Theorem 2.2.

Proof. We wish to prove that every convergent sequence is bounded. Suppose

$$\lim_{n\to\infty} x_n = L.$$

Then, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - L| < \varepsilon$. Set $\varepsilon = 1$. Then, take $K \in \mathbb{N}$ such that $|x_n - L| < 1$ for all $n \geq K$. So, $L-1 < x_n < L+1$. Let $|x_n| = \max\{|L-1|, |L+1|\}$ for all $n \geq K$. Since $\{|x_1|, \ldots, |x_{K-1}|\}$ is a finite set of numbers in F, it is bounded, so it contains a maximum. As such, for $1 \leq n \leq K-1$, we have $|x_n| \leq A$ for some $A \in F$. Define

$$M = \max\{|L-1|, |L+1|, A\}$$

so $|x_n| \leq M$ and the result follows. □

Example 2.16 (sequences in $\mathbb{Q}$ diverge)**.** Let $x_n = n$. Then, the sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{Q}$ is not bounded in $\mathbb{Q}$ (simple application of the Archimedean property in $\mathbb{Q}$). By **(i)** of Theorem 2.2, $\{x_n\}_{n\in\mathbb{N}}$ does not converge in $\mathbb{Q}$.

We then prove **(ii)** of Theorem 2.2.

Proof. We shall prove that

$$\lim_{n\to\infty} (x_n + y_n) = L_1 + L_2.$$

We know that there exist $K_1, K_2 \in \mathbb{N}$ such that

$$|x_n - L_1| < \frac{\varepsilon}{2} \text{ for all } n \geq K_1 \quad \text{and} \quad |y_n - L_2| < \frac{\varepsilon}{2} \text{ for all } n \geq K_2.$$

Set $K = \max\{K_1, K_2\}$. By the triangle inequality (Theorem 1.8),

$$|x_n - L_1 + y_n - L_2| < |x_n - L_1| + |y_n - L_2| < \varepsilon$$

and the result follows. □

For **(iii)** of Theorem 2.2, we only prove the result involving the product of two sequences.

Proof. Since $|x_n|$ is convergent, then it is bounded by **(i)** of Theorem 2.2, i.e. $|x_n| \leq M_1$ for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned} |x_n y_n - L_1 L_2| &= |x_n y_n - x_n L_2 + x_n L_2 - L_1 L_2| \\ &\leq |x_n y_n - x_n L_2| + |x_n L_2 - L_1 L_2| \quad \text{by the triangle inequality (Theorem 1.8)} \\ &= |x_n||y_n - L_2| + |L_2||x_n - L_1| \\ &\leq M_1 |y_n - L_2| + |L_2||x_n - L_1| \end{aligned}$$

Set $M = \max\{M_1, |L_2|\} > 0$. So,

$$M_1|y_n - L_2| + |L_2||x_n - L_1| \leq M(|y_n - L_2| + |x_n - L_1|).$$

Let $\varepsilon > 0$ be arbitrary. Then, there exist $K_1, K_2 \in \mathbb{N}$ such that

$$\begin{aligned} |x_n - L_1| &< \varepsilon/2M \text{ for all } n \geq K_1 \\ |y_n - L_2| &< \varepsilon/2M \text{ for all } n \geq K_2 \end{aligned}$$

Let $K = \max\{K_1, K_2\}$. Hence,

$$|x_n y_n - L_1 L_2| < M\left(\frac{\varepsilon}{2M} + \frac{\varepsilon}{2M}\right) < \varepsilon.$$

and we are done. □

Example 2.17 (Bartle and Sherbert p. 62 Question 10)**.** Prove that if

$$\lim_{n\to\infty} x_n = x > 0,$$

then there exists $M \in \mathbb{N}$ such that $x_n > 0$ for all $n \geq M$.

Solution. By the formal definition of a limit, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - x| < \varepsilon$. Hence,

$$x - \varepsilon < x_n < x + \varepsilon.$$

Since $\varepsilon > 0$ can be made sufficiently small, we can let $\varepsilon = x/2$ so $x_n > x/2 > 0$. Choosing $M = N$, the result follows. □

Example 2.18 (Bartle and Sherbert p. 62 Question 18)**.** If

$$\lim_{n\to\infty} x_n = x > 0,$$

show that there exists $K \in \mathbb{N}$ such that if $n \geq K$, then $x/2 < x_n < 2x$.

Solution. Let $\varepsilon > 0$ be arbitrary. Then, there exists $K \in \mathbb{N}$ such that for all $n \geq K$, we have $|x_n - x| < \varepsilon$. So, $x - \varepsilon < x_n < x + \varepsilon$. Since $\varepsilon > 0$ can be made sufficiently small, then we can let $\varepsilon = x/2$ so $x/2 < x_n < 3x/2 < 2x$. □

Corollary 2.1. If x_n converges and $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n\right)^k.$$

Theorem 2.3 (squeeze theorem)**.** Let x_n, y_n and z_n be sequences of numbers such that for all $n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$. If

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L \quad \text{then} \quad \lim_{n\to\infty} y_n = L.$$

Proof. Let $\varepsilon > 0$. Then, there exists $K \in \mathbb{N}$ such that for all $n \geq K$, we have

$$|x_n - a| < \varepsilon \quad \text{and} \quad |z_n - a| < \varepsilon.$$

Working with the modulus, we have

$$-\varepsilon < x_n - a < \varepsilon \quad \text{and} \quad -\varepsilon < z_n - a < \varepsilon.$$

Thus,

$$-\varepsilon < x_n - a \leq y_n - a \leq z_n - a < \varepsilon$$

which implies $|y_n - a| < \varepsilon$. □

Example 2.19. Evaluate the following limit:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$$

Even though one might think that the Riemann sum comes into play, it actually does not work in this case because

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{1+k/n^2}}$$

and setting

$$f\left(\frac{k}{n}\right) = \sqrt{1+\frac{k}{n^2}},$$

it is impossible to obtain an explicit expression for $f(x)$.

Solution. We use the squeeze theorem to help us. As

$$\frac{n}{\sqrt{n^2+n}} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{n^2}},$$

then

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \leq \frac{n}{\sqrt{n^2}} \\
\lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \leq 1 \\
1 &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \leq 1
\end{aligned}$$

By the squeeze theorem, the required limit is 1. □

Theorem 2.4 (limit theorems)**.** The following hold:

(i) For any $p, q \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \frac{1}{n^{p/q}} = 0$$

(ii) For any $p > 0$, we have

$$\lim_{n\to\infty} \sqrt[n]{p} = 1$$

(iii) We have

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

(iv) For any $a > 1$ and $k \in \mathbb{Z}_{\geq 0}$ sufficiently large, we have

$$\lim_{n\to\infty} \frac{n^k}{a^n} = 0$$

(v) For any $x \in \mathbb{R}$ with $|x| < 1$, one has

$$\lim_{n\to\infty} x^n = 0$$

We first prove **(i)** of Theorem 2.4.

Proof. Given any $\varepsilon > 0$, by Theorem 1.2, there exists a unique $\left(\varepsilon^{1/p}\right)^q > 0$ such that $\left(\varepsilon^{q/p}\right)^p = \varepsilon^q$. By the Archimedean property (Proposition 1.10), there exists $N \in \mathbb{N}$ such that $N \cdot \varepsilon^{q/p} > 1$. Thus, for all $n \geq N$,

$$n \cdot \varepsilon^{q/p} > 1 \quad \text{so} \quad n^p \cdot \varepsilon^q > 1.$$

As such,

$$0 < \frac{1}{n^p} < \varepsilon^q \quad \text{so} \quad 0 < \frac{1}{n^{p/q}} < \varepsilon^{1/q}.$$

Hence, the result follows. □

We then prove **(ii)** of Theorem 2.4.

Proof. There are three cases to consider. Firstly, if $p = 1$, then we obtain the constant sequence 1, so obviously the limit is 1 as well. Next, if $p > 1$, for every $n \in \mathbb{N}$, set $x_n = \sqrt[n]{p} - 1$. Then, $p = (1 + x_n)^n$. By the binomial theorem (one can also interpret it as Bernoulli's inequality in Theorem 1.5),

$$\text{for all } n \in \mathbb{N} \quad \text{we have} \quad p \geq 1 + nx_n \text{ so } 0 \leq x_n \leq \frac{p-1}{n}.$$

By the squeeze theorem 2.3, the result follows.

For the case where $p < 1$, then $1/p > 1$ so

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{p}} = 1.$$

Hence,

$$\lim_{n\to\infty} \sqrt[n]{p} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{1/p}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{1/p}} = \frac{1}{1} = 1.$$

The result follows. □

Next, we prove **(iii)** of Theorem 2.4.

Proof. For each $n \in \mathbb{N}$, set $x_n = \sqrt[n]{n} - 1$, so $n = (1 + x_n)^n$. By Bernoulli's inequality (Theorem 1.5), for all $n \geq N$, we have

$$n \geq 1 + nx_n \quad \text{so} \quad 0 < x_n \leq \frac{n-1}{n} = 1 - \frac{1}{n}$$

but this is a *useless* statement because it just shows that $0 < x_n \leq 1$. Sadly, we are unable to apply the squeeze theorem here. As such, we use the binomial theorem. Observe that for $n \geq 2$, we have

$$n = (1 + x_n)^n = 1 + nx_n + \binom{n}{2} x_n^2 + \ldots + x_n^n \geq \frac{n(n-1)}{2} x_n^2.$$

As such, for $n \geq 2$, we have

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}.$$

By the squeeze theorem (Theorem 2.3), the limit of x_n is 0, so the limit of $\sqrt[n]{n}$ is 1. □

We then prove **(iv)** and **(v)** of Theorem 2.4.

Proof. Write $a = 1 + p$ with $p > 0$. Consider $n \in \mathbb{N}$ with $n > 2k$. Then, we have

$$a^n = (1+p)^n = \sum_{k=0}^{n} \binom{n}{k} p^k > \binom{n}{k} p^k$$

where we used the binomial theorem. Upon *expansion*, the above is equal to

$$\frac{n(n-1)\dots(n-k+1)}{k!} \cdot p^k > \left(\frac{n}{2}\right)^k \cdot \frac{p^k}{k!}.$$

Hence,

$$0 < \frac{n^k}{a^n} < \frac{2^k k!}{p^k}.$$

However, the RHS is some constant (independent of n), so similar to our proof of **(iii)** of Theorem 2.4, we run into an error. So, we take a detour and consider

$$a^n = (1+p)^n = \sum_{k=0}^{n} \binom{n}{k} p^k > \binom{n}{k+1} p^{k+1} = \frac{n(n-1)\dots(n-k+1)(n-k)}{k!(k+1)} \cdot p^{k+1} > \left(\frac{n}{2}\right)^{k+1} \cdot \frac{p^{k+1}}{(k+1)!}.$$

Hence,

$$0 < \frac{n^k}{a^n} < \frac{2^{k+1}(k+1)!}{p^{k+1}} \cdot \frac{1}{n}.$$

By the squeeze theorem, the original limit is equal to 0. **(v)** follows from **(iv)** by setting $k = 0$ and $a = 1/|x|$. □

Example 2.20 (Bartle and Sherbert p. 62 Question 15)**.** Show that

$$\lim_{n\to\infty} (2n)^{1/n} = 1.$$

Solution. Recall **(iii)** of Theorem 2.4, where it was mentioned that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

Hence,

$$\lim_{n\to\infty} (2n)^{1/n} = \lim_{n\to\infty} 2^{1/n} \cdot \lim_{n\to\infty} n^{1/n} = 1 \cdot 1 = 1.$$

□

Example 2.21 (Bartle and Sherbert p. 62 Question 16)**.** Show that

$$\lim_{n\to\infty} \frac{n^2}{n!} = 0.$$

Solution. We have

$$\lim_{n\to\infty} \frac{n^2}{n!} = \lim_{n\to\infty} \frac{n^2}{n(n-1)} \cdot \lim_{n\to\infty} \frac{1}{(n-2)!} = 1 \cdot 0 = 0.$$

By the squeeze theorem (Theorem 2.3), the result follows. □

Example 2.22 (Bartle and Sherbert p. 62 Question 17)**.** Show that

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

Hint: If $n \geq 3$, then $0 < \frac{2^n}{n!} \leq 2\left(\frac{2}{3}\right)^{n-2}$

Solution. If $n \geq 3$, then $n! \geq 3^{n-2}$. As such,

$$0 = \lim_{n\to\infty} 0 \leq \lim_{n\to\infty} \frac{2^n}{n!} \leq \lim_{n\to\infty} \frac{2^n}{3^{n-2}} = 0.$$

By the squeeze theorem (Theorem 2.3), the result follows. □

Example 2.23 (Bartle and Sherbert p. 70 Question 14). Determine the limits of the following sequences:

(a) $\lim_{n\to\infty} n^{1/n^2}$

(b) $\lim_{n\to\infty} (n!)^{1/n^2}$

Solution.

(a) Let $x_n = n^{1/n^2} - 1$. Then, $(x_n + 1)^{n^2} = n$. By the binomial theorem, we have

$$\sum_{k=0}^{n^2} \binom{n^2}{k} x_n^k = n \quad \text{so} \quad n \geq 1 + n^2 x_n.$$

As such,

$$x_n \leq \frac{1}{n} - \frac{1}{n^2} \quad \text{which implies} \quad \lim_{n\to\infty} x_n \leq 0.$$

As for the lower bound, note that for $n \geq 2$, we have $x_n \geq 0$, so by **(iv)** of Theorem 2.2,

$$\lim_{n\to\infty} x_n \geq 0.$$

Combining both inequalities shows that the limit of x_n is 0, so the original limit is 1.

(b) We note that for $n \geq 4$, we have

$$n^2 \leq n! \leq n^n$$

so

$$\lim_{n\to\infty} \left(n^2\right)^{1/n^2} \leq \lim_{n\to\infty} (n!)^{1/n^2} \leq \lim_{n\to\infty} (n^n)^{1/n^2} \quad \text{or equivalently} \quad \left(\lim_{n\to\infty} n^{1/n^2}\right)^2 \leq \lim_{n\to\infty} (n!)^{1/n^2} \leq \lim_{n\to\infty} n^{1/n}.$$

By **(a)**, the lower bound is 1 and by **(iii)** of Theorem 2.4, the upper bound is 1. Hence, by the squeeze theorem (Theorem 2.3), the desired limit is 1. □

Theorem 2.5 (limit theorems). The following hold:

(i) If

$$\lim_{n\to\infty} |x_n| = 0 \quad \text{then} \quad \lim_{n\to\infty} x_n = 0.$$

If $c = n$, the limit is still the same.

(ii) If

$$\lim_{n\to\infty} x_n = L \quad \text{then} \quad \lim_{n\to\infty} |x_n| = |L|.$$

(iii) Suppose $x_n \geq 0$ for all $n \in \mathbb{N}$. Then,

$$\lim_{n\to\infty} x_n = L \quad \text{implies} \quad \lim_{n\to\infty} \sqrt{x_n} = \sqrt{L}.$$

(iv) If $x_n \geq 0$ for all $n \in \mathbb{N}$ and x_n converges, then

$$\lim_{n\to\infty} x_n \geq 0.$$

To see an application/proof of **(ii)** of Theorem 2.5 as well as its reverse direction, see Example 2.24.

Example 2.24 (Bartle and Sherbert p. 62 Question 8). Prove that

$$\lim_{n\to\infty} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} |x_n| = 0.$$

Give an example to show that the convergence of $\{|x_n|\}_{n\in\mathbb{N}}$ need not imply the convergence of $\{x_n\}_{n\in\mathbb{N}}$.

Solution. For the first part, we first prove the forward direction. Suppose $x_n \to 0$. Then, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n| < \varepsilon$. Since applying the absolute value function twice is the same as applying it once, the forward direction holds.

For the proof of the reverse direction, suppose $|x_n| \to 0$. Then, for all $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $n \geq K$, we have $||x_n| - 0| < \varepsilon$. Same as the reasoning provided earlier, the reverse direction holds.

For the second part, let $x_n = (-1)^n$. Then, $\{|x_n|\}_{n\in\mathbb{N}}$ converges because $|x_n| = 1$ which is the constant sequence 1 but by Example 2.15, $\{x_n\}_{n\in\mathbb{N}}$ is not convergent. $\square$

Example 2.25 (Bartle and Sherbert p. 70 Question 7)**.** If $\{b_n\}_{n\in\mathbb{N}}$ is a bounded sequence and $\lim_{n\to\infty} a_n = 0$, show that

$$\lim_{n\to\infty} a_n b_n = 0.$$

Solution. Since b_n is bounded, then for all $n \in \mathbb{N}$, there exists $M \in \mathbb{R}$ such that $-M \leq b_n \leq M$. As such,

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) \quad \text{under the assumption that both limits exist}$$

Note that

$$0 = 0 \cdot (-M) \leq \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) \leq 0 \cdot M = 0$$

so by the squeeze theorem (Theorem 2.3), the result follows. $\square$

Alternatively, we can prove the result in Example 2.25 more formally.

Solution. Since

$$\lim_{n\to\infty} a_n = 0,$$

then for every $\varepsilon, M > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n| < \varepsilon/M$. Again, since b_n is bounded, then for all $n \in \mathbb{N}$, there exists $M \in \mathbb{R}^+$ such that $-M \leq b_n \leq M$. So, $(-\varepsilon/M) \cdot M \leq a_n b_n \leq (\varepsilon/M) \cdot M$. Since ε can be made sufficiently small, by the squeeze theorem (Theorem 2.3), the result follows. $\square$

Example 2.26 (Bartle and Sherbert p. 70 Question 20)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} x_n^{1/n} = L < 1.$$

Show that there exists a number r with $0 < r < 1$ such that $0 < x_n < r^n$ for all sufficiently large $n \in \mathbb{N}$. Use this to show that

$$\lim_{n\to\infty} x_n = 0.$$

Solution. By the formal definition of a limit, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\left|x_n^{1/n} - L\right| < \varepsilon$. We choose $\varepsilon = (1-L)/2$. Then,

$$L - \frac{1-L}{2} < x_n^{1/n} < L + \frac{1-L}{2} \quad \text{so} \quad \frac{3L-1}{2} < x_n^{1/n} < \frac{L+1}{2}.$$

Raising each side to the power n yields the inequality

$$\left(\frac{3L-1}{2}\right)^n < x_n < \left(\frac{L+1}{2}\right)^n.$$

We can choose $r = (L+1)/2$. By the squeeze theorem, the result follows. $\square$

Corollary 2.2. If $a, b \in \mathbb{R}$ and $a \le x_n \le b$ for all $n \in \mathbb{N}$ and x_n is convergent, then

$$a \le \lim_{n\to\infty} x_n \le b.$$

Example 2.27. Suppose we wish to evaluate the following limit:

$$\lim_{n\to\infty} \frac{2^n + 3^{n+1} + 5^{n+2}}{2^{n+2} + 3^n + 5^{n+1}}$$

Solution. Recognise that for $0 \le a < 1$, then $a^n \to 0$ as $n \to \infty$.

$$\begin{aligned}\lim_{n\to\infty} \frac{2^n + 3^{n+1} + 5^{n+2}}{2^{n+2} + 3^n + 5^{n+1}} &= \lim_{n\to\infty} \frac{2^n + 3\left(3^n\right) + 25\left(5^n\right)}{4\left(2^n\right) + 3^n + 5\left(5^n\right)} \\ &= 5 - \lim_{n\to\infty} \frac{19\left(2^n\right) + 2\left(3^n\right)}{4\left(2^n\right) + 3^n + 5\left(5^n\right)} \\ &= 5 - \lim_{n\to\infty} \frac{19\left(\frac{2}{5}\right)^n + 2\left(\frac{3}{5}\right)^n}{4\left(\frac{2}{5}\right)^n + \left(\frac{3}{5}\right)^n + 5} \\ &= 5\end{aligned}$$

□

Example 2.28 (Bartle and Sherbert p. 70 Question 12)**.** If $0 < a < b$, determine

$$\lim_{n\to\infty} \frac{a^{n+1} + b^{n+1}}{a^n + b^n}.$$

Solution. We have

$$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = b \cdot \frac{(a/b)^{n+1} + 1}{(a/b)^n + 1}$$

so the limit evaluates to b. □

Example 2.29 (Bartle and Sherbert p. 84 Question 5)**.** Let $X = x_n$ and $Y = y_n$ be given sequences, and let the "shuffled" sequence $Z = z_n$ be defined by

$$z_1 = x_1, z_2 = y_1, \ldots, z_{2n-1} = x_n, z_{2n} = y_n.$$

Show that

$$Z \text{ is convergent} \quad \text{if and only if} \quad X \text{ and } Y \text{ are convergent and } \lim_{n\to\infty} X = \lim_{n\to\infty} Y.^{\dagger}$$

Solution. We first prove the reverse direction. Suppose X and Y are convergent and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = L.$$

By the definition of a limit of a sequence, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - L| < \varepsilon \text{ whenever } n \ge N_1 \quad \text{and} \quad |y_n - L| < \varepsilon \text{ whenever } n \ge N_2.$$

Set $N = \max\{2N_1, 2N_2\}$. Then, whenever $n \ge N$, $|z_n - L| < \varepsilon$ and we are done.

Now, we prove the reverse direction. Suppose Z is convergent. That is

$$\lim_{n\to\infty} z_n = L.$$

†Refer to this problem on StackExchange here.

By the definition of the limit of a sequence, there exists $N \in \mathbb{N}$ such that

$$|z_n - L| < \varepsilon \text{ whenever } n \geq N.$$

We need to show that

$$|x_n - L| < \varepsilon \text{ and } |y_n - L| < \varepsilon \text{ whenever } n \geq N,$$

which are

$$|z_{2n-1} - L| < \varepsilon \text{ and } |z_{2n} - L| < \varepsilon \quad \text{equivalently}.$$

Thus, we need $2n-1 \geq N$ and $2n \geq N$, which are obviously true. Hence, the result follows. □

Example 2.30 (MA2108 AY19/20 Sem 1). Let $f : (a,\infty) \to \mathbb{R}$ be a function such that it is bounded in any interval (a,b) and

$$\lim_{x\to\infty} (f(x+1) - f(x)) = A.$$

Prove that

$$\lim_{x\to\infty} \frac{f(x)}{x} = A.$$

Solution. Let $\varepsilon > 0$ be arbitrary. By the given limit, there exists $M > 0$ such that for all $x > M$,

$$|f(x+1) - f(x) - A| < \varepsilon.$$

So,

$$A - \varepsilon < f(x+1) - f(x) < A + \varepsilon.$$

Since f is locally bounded, then for $M < x \leq M+1$, $-B < f(x) < B$ for some $B \in \mathbb{R}$. Hence,

$$-B + (A+\varepsilon) \cdot \lfloor x - M \rfloor < f(x) < B + (A+\varepsilon) \cdot \lceil x - M \rceil.$$

Dividing by x on both sides, since ε is made arbitrarily small, by the squeeze theorem, $f(x)/x$ tends to A as $x \to \infty$. □

Theorem 2.6 (L'Hôpital's Rule). If f and g are differentiable functions such that $g'(x) \neq 0$ on an open interval I containing a,

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

and

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} \text{ exists} \quad \text{then} \quad \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Theorem 2.7 (Stolz-Cesàro theorem). Let x_n and y_n be two sequences of real numbers. If y_n is strictly monotone and divergent and

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = L \text{ exists} \quad \text{then} \quad \lim_{n\to\infty} \frac{x_n}{y_n} = L.$$

We will give a proof of the Stolz-Cesàro theorem (Theorem 2.7) in Example 2.31, where without loss of generality, we assume that the sequence $\{b_n\}_{n\in\mathbb{N}}$ (in place of $\{y_n\}_{n\in\mathbb{N}}$ in Theorem 2.7) is monotonically increasing.

Example 2.31 (MA2108S AY24/25 Sem 2 Tutorial 3; Stolz-Cesàro). Let $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ be sequences, where $\{b_n\}_{n\in\mathbb{N}}$ is strictly increasing and divergent. Prove that

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = A \quad \text{implies} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = A.$$

Solution. By the formal definition of a limit, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left|\frac{a_{n+1} - a_n}{b_{n+1} - b_n} - A\right| < \varepsilon.$$

So,

$$(A - \varepsilon)(b_{n+1} - b_n) < a_{n+1} - a_n < (A + \varepsilon)(b_{n+1} - b_n)$$

By the method of difference,

$$\sum_{n=N}^{k-1} (A - \varepsilon)(b_{n+1} - b_n) < \sum_{n=N}^{k-1} a_{n+1} - a_n < \sum_{n=N}^{k-1} (A + \varepsilon)(b_{n+1} - b_n)$$

so

$$(A - \varepsilon)(b_k - b_N) < a_k - a_N < (A + \varepsilon)(b_k - b_N).$$

Adding a_N to each side yields

$$(A - \varepsilon)(b_k - b_N) + a_N < a_k < (A + \varepsilon)(b_k - b_N) + a_N.$$

For k sufficiently large, we have $b_k > 1$ so $1/b_k < 1$. As such,

$$\frac{(A - \varepsilon)(b_k - b_N) + a_N}{b_k} < \frac{a_k}{b_k} < \frac{(A + \varepsilon)(b_k - b_N) + a_N}{b_k}.$$

So,

$$(A - \varepsilon)\left(1 - \frac{b_N}{b_k}\right) + \frac{a_N}{b_k} < \frac{a_k}{b_k} < (A + \varepsilon)\left(1 - \frac{b_N}{b_k}\right) + \frac{a_N}{b_k}.$$

Letting $k \to \infty$, we see that a_k/b_k is sandwiched between A and A, so by the squeeze theorem (Theorem 2.3), the result follows. □

Theorem 2.8 (Stolz-Cesàro theorem, alt.). If

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$$

where y_n is strictly decreasing and

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = L \quad \text{then} \quad \lim_{n\to\infty} \frac{x_n}{y_n} = L.$$

Example 2.32 (MA2108 AY19/20 Sem 1). Let a_n be a sequence in $\mathbb{R}$.

(i) Prove that if

$$\lim_{n\to\infty} a_n = a \quad \text{then} \quad \lim_{n\to\infty} \frac{a_1 + a_2 + \ldots + a_n}{n} = a.$$

(ii) Suppose the sequence

$$\frac{a_1 + a_2 + \ldots + a_n}{n}$$

converges. Can we deduce that a_n converges? Justify your answer.

Solution.

(i) Let $\varepsilon > 0$ be arbitrary. There exists $K_1 \in \mathbb{N}$ such that $|a_j - a| \le \varepsilon/2$ for all $j \ge K_1$. Then,

$$\left|\frac{a_1 + a_2 + \ldots + a_n}{n} - a\right| = \left|\frac{1}{n}\sum_{j=1}^{n}(a_j - a)\right|.$$

We can bound this sum accordingly. For $n \ge K_1$,

$$\begin{aligned}
\left|\frac{1}{n}\sum_{j=1}^{n}(a_j - a)\right| &\le \frac{1}{n}\left|\sum_{j=1}^{K_1}(a_j - a)\right| + \frac{1}{n}\left|\sum_{j=K_1+1}^{n}(a_j - a)\right| && \text{by triangle inequality} \\
&\le \frac{1}{n}\left|\sum_{j=1}^{K_1}(a_j - a)\right| + \frac{n-K_1}{n}\cdot\frac{\varepsilon}{2} \\
&< \frac{1}{n}\left|\sum_{j=1}^{K_1}(a_j - a)\right| + \frac{\varepsilon}{2} \\
&= \frac{C}{n} + \frac{\varepsilon}{2}
\end{aligned}$$

Here, we let C be the sum of $a_j - a$ from $j = 1$ to $j = K_1$. Next, for $K \in \mathbb{N}$, where $K > \max\{K_1, 2C/\varepsilon\}$, it is now easy to see that

$$\begin{aligned}
\left|\frac{1}{n}\sum_{j=1}^{n}(a_j - a)\right| &< \frac{C}{n} + \frac{\varepsilon}{2} \\
&\le \frac{C}{K} + \frac{\varepsilon}{2} && \text{since } n \ge K \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{since } K > \frac{2C}{\varepsilon} \\
&= \varepsilon
\end{aligned}$$

(ii) No. Define $s_n = (a_1 + a_2 + \ldots + a_n)/n$. Setting $a_n = (-1)^n$,

$$s_n = \begin{cases} -1/n & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

By the squeeze theorem, as $n \to \infty$, $s_n \to 0$, so it converges. However, a_n diverges. □

2.2 Monotone Sequences

Definition 2.7 (monotone sequence). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that it is

(i) monotonically increasing if $x_n \le x_{n+1}$ for all $n \in \mathbb{N}$;

(ii) monotonically decreasing if $x_n \ge x_{n+1}$ for all $n \in \mathbb{N}$

Proposition 2.2. Let F be an ordered field. If F has the least upper bound property, then

every monotone sequence in F	is	bounded
every monotone increasing sequence in F	is	bounded above
every monotone decreasing sequence in F	is	bounded below

Theorem 2.9 (monotone convergence theorem). Let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone sequence. Then,

$$\{x_n\}_{n\in\mathbb{N}} \text{ converges} \quad \text{if and only if} \quad \text{it is bounded.}$$

In particular, if

$$x_n \text{ is increasing then } \lim_{n\to\infty} x_n = \sup x_n \quad \text{and if} \quad x_n \text{ is decreasing then } \lim_{n\to\infty} x_n = \inf x_n.$$

Proof. It suffices to show that if $\{x_n\}_{n\in\mathbb{N}}$ is a monotonically increasing sequence in F which is bounded above, then there exists $x \in F$ such that $x_n \to x$ in F. Let

$$S = \{x_n \in F : n \in \mathbb{N}\} = \{x \in F : \text{there exists } n \in \mathbb{N} \text{ such that } x = px_n\}$$

denote the image set of the sequence $\{x_n\}_{n\in\mathbb{N}}$. Since $\mathbb{N} \neq \emptyset$, then $S \neq \emptyset$. As $\{x_n\}_{n\in\mathbb{N}}$ is bounded above, then $S \subseteq F$ is also bounded above.

By the least upper bound property of F (Definition 1.10), there exists $x = \sup S$ in F. We claim that $x_n \to x$ in F, i.e.

$$\text{for any } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \quad \text{such that for all } n \geq N \text{ we have } |x_n - x| < \varepsilon.$$

Since $\varepsilon > 0$, then $x - \varepsilon \in F$ is not an upper bound of S. So, there exists $x' \in S$ such that $x - \varepsilon < x'$, i.e. there exists $N \in \mathbb{N}$ such that $x - \varepsilon < x_N$. However, as $x = \sup S$ is an upper bound of S, then $x_n \leq x$. As $\{x_n\}_{n\in\mathbb{N}}$ is monotonically increasing, we know that

$$\text{for all } n \geq N \quad \text{we have} \quad x_N \leq x_n.$$

Equivalently, we have $x - \varepsilon < x_n \leq x$, so $|x - x_n| < \varepsilon$. We conclude that x_n tends to $\sup x_n$. □

Example 2.33 (MA2108S AY16/17 Sem 2 Homework 4). Let $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that x_n converges and find the limit.

Solution. It is clear that x_n is bounded above by 2. Given that $x_1 = 1$, we show that x_n is strictly increasing. That is, for $n \in \mathbb{N}$, $x_{n+1} > x_n$.

$$x_{n+1} - x_n = \sqrt{2 + x_n} - x_n = \frac{2 + x_n - x_n^2}{\sqrt{2 + x_n} + x_n} = \frac{(x_n - 2)(x_n + 1)}{\sqrt{2 + x_n} + x_n}.$$

It is clear that x_n is a sequence of positive terms so we consider the numerator of $x_{n+1} - x_n$, which is $(2 - x_n)(x_n + 1)$. For $1 \leq x_n \leq 2$, this product is always positive, and hence $x_{n+1} - x_n \geq 0$. By the monotone convergence theorem (Theorem 2.9), x_n converges.

Suppose

$$\lim_{n\to\infty} x_n = L.$$

Then, $L = \sqrt{2 + L}$, but since $L > 0$, then $L = 2$. □

Example 2.34. Consider the recurrence relation

$$a_{n+1} = \frac{3 + a_n}{1 + a_n} \quad \text{with the initial condition} \quad a_1 = 3.$$

Prove that a_n is a convergent sequence and find its limit.

Solution. We first prove that $\sqrt{3} \le a_n \le 3$. To show that $a_n \le 3$, we have

$$\begin{aligned} a_{k+1} = \frac{3+a_k}{1+a_k} &\le \frac{3+3}{1+\sqrt{3}} \quad \text{by the induction hypothesis} \\ &\le 3 \end{aligned}$$

Similarly, we have

$$a_{k+1} = \frac{3+a_k}{1+a_k} \ge \frac{3+\sqrt{3}}{1+3} \ge \sqrt{3}$$

where again, the first inequality follows by the induction hypothesis. This shows that a_n is bounded.

We then prove that a_n is decreasing using strong induction. We have

$$a_{k+1} - a_k = \frac{3+a_k}{1+a_k} - a_k = \frac{3-a_k^2}{1+a_k}.$$

It suffices to prove that $3 - a_k^2 \le 0$ since the denominator $1+a_k > 0$. Since $\sqrt{3} \le a_k \le 3$, then $3 \le a_k^2 \le 9$ so $3 - a_k^2 \le 0$. So, a_n is decreasing. By the monotone convergence theorem, a_n converges to some limit L. Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L \quad \text{then} \quad L = \frac{3+L}{1+L}.$$

So, either $L = \sqrt{3}$ or $L = -\sqrt{3}$. We reject the latter as we earlier established that a_n is a sequence of positive numbers (from $\sqrt{3} \le a_n \le 3$) so $L = 3$. □

Example 2.35 (Bartle and Sherbert p. 70 Question 9)**.** Let

$$y_n = \sqrt{n+1} - \sqrt{n} \quad \text{for } n \in \mathbb{N}.$$

Show that $\{\sqrt{n} y_n\}_{n\in\mathbb{N}}$ converges. Find the limit.

Solution. Let $x_n = \sqrt{n} y_n$. Then,

$$x_n = \sqrt{n(n+1)} - n.$$

We first prove that $\{x_n\}_{n\in\mathbb{N}}$ is bounded, i.e. $0 \le x_n \le 1/2$. Proving the lower bound is obvious because it is equivalent to showing that

$$n(n+1) \ge n^2 \quad \text{or equivalently} \quad n \ge 0.$$

The aforementioned statement holds trivially. We then justify the upper bound, i.e.

$$\sqrt{n(n+1)} - n - \frac{1}{2} \le 0 \quad \text{or equivalently} \quad n(n+1) \le \left(n + \frac{1}{2}\right)^2.$$

We have

$$n^2 + n \le n^2 + n + \frac{1}{4} \quad \text{or equivalently} \quad \frac{1}{4} \ge 0.$$

Hence, $\{x_n\}_{n\in\mathbb{N}}$ is bounded.

Next, we prove that x_n is increasing by induction. We have

$$\begin{aligned} x_{k+1} - x_k &= \sqrt{(k+1)(k+2)} - (k+1) - \sqrt{k(k+1)} + k \\ &= \sqrt{k+1}\left(\sqrt{k+2} - \sqrt{k}\right) - 1 \\ &= \frac{2\sqrt{k+1} - \sqrt{k+2} - \sqrt{k}}{\sqrt{k+2} + \sqrt{k}} \end{aligned}$$

As such, it suffices to prove that $2\sqrt{k+1}-\sqrt{k+2}-\sqrt{k}\geq 0$. To see why this holds, define $z_k=\sqrt{k+1}-\sqrt{k}$. Then, the mentioned inequality is equivalent to $z_k-z_{k+1}\geq 0$, or $z_{k+1}\leq z_k$. As it is known that z_k is a decreasing sequence, then $x_{k+1}\geq x_k$, i.e. x_n is increasing. By the monotone convergence theorem (Theorem 2.9), $\{x_n\}_{n\in\mathbb{N}}$ converges.

Hence,

$$\lim_{n\to\infty} x_n=\lim_{n\to\infty}\frac{n(n+1)-n^2}{\sqrt{n(n+1)}+n}=\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}+n}=\lim_{n\to\infty}\frac{1}{\sqrt{1+1/n}+1}=\frac{1}{2}.$$

□

Example 2.36 (Bartle and Sherbert p. 77 Question 2). Let $x_1>1$ and

$$x_{n+1}=2-\frac{1}{x_n}\quad\text{for } n\in\mathbb{N}.$$

Show that $\{x_n\}_{n\in\mathbb{N}}$ is bounded and monotone. Find the limit.

Solution. We first prove that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. We claim that $x_n>1$ for all $n\in\mathbb{N}$. The base case holds trivially. Assume that $x_k>1$ for some $k\in\mathbb{N}$. Then, $-1/x_k>-1$ so $x_{k+1}>1$. As such, $\{x_n\}_{n\in\mathbb{N}}$ is bounded below by induction.

We then prove that $\{x_n\}_{n\in\mathbb{N}}$ is monotonically decreasing. Assume that $x_{k+1}-x_k\leq 0$ for all $k\leq n-1$. Then,

$$x_{n+1}-x_n=2-\frac{1}{x_n}-x_n=\frac{2x_n-x_n^2-1}{x_n}=-\frac{(x_n-1)^2}{x_n}.$$

Since x_n is a sequence of positive numbers (we deduce that x_n is bounded below by 1 earlier), then $x_{n+1}-x_n<0$, so $x_{n+1}<x_n$. By induction, $\{x_n\}_{n\in\mathbb{N}}$ is monotonically decreasing. Hence, $\{x_n\}_{n\in\mathbb{N}}$ converges. Suppose the limit is L. Then,

$$L=2-\frac{1}{L}.$$

Hence, $L=1$. □

Example 2.37 (Bartle and Sherbert p. 77 Question 3). Let $x_1\geq 2$ and $x_{n+1}=1+\sqrt{x_n-1}$ for $n\in\mathbb{N}$. Show that $\{x_n\}_{n\in\mathbb{N}}$ is decreasing and bounded below by 2. Find the limit.

Solution. We first show that $\{x_n\}_{n\in\mathbb{N}}$ is bounded below by 2. We have

$$x_{n+1}=1+\sqrt{x_n-1}\geq 1+\sqrt{2-1}=2$$

so by induction, $\{x_n\}_{n\in\mathbb{N}}$ is bounded below. We then prove that $\{x_n\}_{n\in\mathbb{N}}$ is decreasing. We have

$$x_{n+1}-x_n=1+\sqrt{x_n-1}-x_n=\sqrt{x_n-1}\left(1-\sqrt{x_n-1}\right).$$

Since $\sqrt{x_n-1}\geq 1$, then it follows that $x_{n+1}-x_n<0$, i.e. $\{x_n\}_{n\in\mathbb{N}}$ is decreasing. By the monotone convergence theorem, $\{x_n\}_{n\in\mathbb{N}}$ converges. Suppose the limit is L. Then, $L=1+\sqrt{L-1}$. As such, $L=2$. □

Example 2.38 (Bartle and Sherbert p. 77 Question 6). Let $a>0$ and let $z_1>0$. Define

$$z_{n+1}=\sqrt{a+z_n}\quad\text{for } n\in\mathbb{N}.$$

Show that $\{z_n\}_{n\in\mathbb{N}}$ converges and find the limit.

Solution. We first observe that $\{z_n\}_{n\in\mathbb{N}}$ is a positive sequence of numbers. Consider the equation $L=\sqrt{a+L}$, which yields $L^2-L-a=0$. The positive root of this quadratic equation is

$$r=\frac{1+\sqrt{1+4a}}{2}.$$

As such, we shall consider three cases as follows:

(i) $z_1<r$

(ii) $z_1=r$

(iii) $z_1>r$

For **(i)**, if $z_1<r$, then we claim that $\{z_n\}_{n\in\mathbb{N}}$ is increasing and bounded above by r. We first prove the latter by induction. The base case holds trivially as we earlier mentioned that $z_1<r$. Suppose $z_k<r$ for some $k\in\mathbb{N}$. Then,

$$z_{k+1}=\sqrt{a+z_k}\le\sqrt{a+r}=r.$$

The last equality holds because it is equivalent to $r^2-r-a=0$. As mentioned, r is a root of the quadratic equation $L^2-L-a=0$, so indeed $\sqrt{a+r}=r$. As such, $\{z_n\}_{n\in\mathbb{N}}$ is bounded above.

We then prove that $\{z_n\}_{n\in\mathbb{N}}$ is increasing. We have

$$z_{n+1}-z_n=\sqrt{a+z_n}-z_n=\frac{\left(\sqrt{a+z_n}-z_n\right)\left(\sqrt{a+z_n}+z_n\right)}{\sqrt{a+z_n}+z_n}=\frac{a+z_n-z_n^2}{\sqrt{a+z_n}+z_n}=-\frac{z_n^2-z_n-a}{\sqrt{a+z_n}+z_n}.$$

By considering the denominator, as z_n is a positive sequence of numbers, then $\sqrt{a+z_n},z_n>0$ so it suffices to prove that $z_n^2-z_n-a<0$. Let

$$r'=\frac{-1-\sqrt{1+4a}}{2}\quad\text{be}\quad\text{the negative root of the quadratic equation } L^2-L-a=0.$$

Then, the solution to the quadratic inequality $z_n^2-z_n-a<0$ is $r'<z_n<r$. As we earlier deduced that $0<z_n<r$, then $z_n<r$ holds so $z_{n+1}-z_n>0$, i.e. $\{z_n\}_{n\in\mathbb{N}}$ is increasing. By the monotone convergence theorem, $\{z_n\}_{n\in\mathbb{N}}$ converges.

For **(ii)**, we have the constant sequence $z_n=\sqrt{a+r}$ so $\{z_n\}_{n\in\mathbb{N}}$ converges.

Lastly, for **(iii)**, if $z_1>r$, we claim that $\{z_n\}_{n\in\mathbb{N}}$ is decreasing and bounded below by r. We first prove the latter by induction. Again, the base case holds trivially as we earlier mentioned that $z_1>r$. Suppose $z_k>r$ for some $k\in\mathbb{N}$. Then,

$$z_{k+1}=\sqrt{a+z_k}\ge\sqrt{a+r}=r.$$

Again, the last equality holds due to the same argument made previously, i.e. r is the positive root of the quadratic equation $L^2-L-a=0$. As such, $\{z_n\}_{n\in\mathbb{N}}$ is bounded below.

We then prove that $\{z_n\}_{n\in\mathbb{N}}$ is decreasing. We have

$$z_{n+1}-z_n=-\frac{z_n^2-z_n-a}{\sqrt{a+z_n}+z_n}.$$

Again, it suffices to consider the numerator $z_n^2-z_n-a$. We wish to prove that it is positive. The solution to the quadratic inequality $z_n^2-z_n-a>0$ is $z_n>r$ or $z_n<-r'$. As we earlier deduced that $z_n>r$, then it follows that $\{z_n\}_{n\in\mathbb{N}}$ is decreasing. By the monotone convergence theorem, $\{z_n\}_{n\in\mathbb{N}}$ converges.

As mentioned, the limit is r, which is equal to

$$\frac{1+\sqrt{1+4a}}{2}.$$

We chose the positive root here because $\{z_n\}_{n\in\mathbb{N}}$ is a positive sequence of numbers. $\square$

Example 2.39 (Bartle and Sherbert p. 77 Question 7). Let $x_1 = a > 0$ and

$$x_{n+1} = x_n + \frac{1}{x_n} \quad \text{for } n \in \mathbb{N}.$$

Determine whether $\{x_n\}_{n\in\mathbb{N}}$ converges or diverges.

Solution. We have

$$x_{n+1}^2 = x_n^2 + \frac{1}{x_n^2} + 2 \quad \text{so} \quad x_{n+1}^2 - x_n^2 = 2 + \frac{1}{x_n^2} > 2.$$

By the method of difference, $x_N^2 - x_1^2 > 2(N-1)$ so

$$x_N > \sqrt{a + 2N - 2}.$$

Hence, $x_N \to \infty$ as $N \to \infty$, which implies $\{x_n\}_{n\in\mathbb{N}}$ diverges. $\square$

Example 2.40 (Bartle and Sherbert p. 77 Question 10). Establish the convergence or the divergence of the sequence (y_n), where

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \quad \text{for } n \in \mathbb{N}.$$

Solution. We first prove that $\{y_n\}_{n\in\mathbb{N}}$ is bounded. We have

$$\frac{1}{2} \le \underbrace{\frac{1}{2n} + \frac{1}{2n} + \ldots + \frac{1}{2n}}_{n \text{ copies}} \le \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \le \underbrace{\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}}_{n \text{ copies}} = 1$$

so $1/2 \le y_n \le 1$ for all $n \in \mathbb{N}$. This shows that $\{y_n\}_{n\in\mathbb{N}}$ is bounded.

We then prove that $\{y_n\}_{n\in\mathbb{N}}$ is increasing. We have

$$y_{n+1} - y_n = \left(\frac{1}{n+1} + \ldots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \ldots + \frac{1}{2n}\right) = \frac{1}{2n+1} + \frac{1}{2n+2} > 0$$

so $\{y_n\}_{n\in\mathbb{N}}$ is increasing. By the monotone convergence theorem, $\{y_n\}_{n\in\mathbb{N}}$ converges. $\square$

Example 2.41 (Bartle and Sherbert p. 77 Question 11). Let

$$x_n = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} \quad \text{for each } n \in \mathbb{N}$$

Show that $\{x_n\}_{n\in\mathbb{N}}$ converges.

Solution. We first show that $\{x_n\}_{n\in\mathbb{N}}$ is increasing. We have

$$x_{n+1} - x_n = \frac{1}{(n+1)^2} > 0$$

so $x_{n+1} > x_n$, so $\{x_n\}_{n\in\mathbb{N}}$ is increasing.

Next, we show that $\{x_n\}_{n\in\mathbb{N}}$. We use the fact $k^2 \geq k(k-1)$ for $k \geq 2$ so

$$\frac{1}{k^2} \leq \frac{1}{k(k-1)}.$$

As such,

$$x_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \sum_{k=2}^{n} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} \leq 2$$

where the last inequality uses the method of difference. By the monotone convergence theorem (Theorem 2.9), $\{x_n\}_{n\in\mathbb{N}}$ converges. □

Example 2.42 (Bartle and Sherbert p. 70 Question 18)**.** Let $X = \{x_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L > 1.$$

Show that X is not a bounded sequence and hence is not convergent.

Solution. By the formal definition of a limit, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon.$$

So,

$$L - \varepsilon < \frac{x_{n+1}}{x_n} < L + \varepsilon.$$

Hence,

$$x_{n+1} > (L-\varepsilon)x_n > (L-\varepsilon)^2 x_{n-1} > \ldots > (L-\varepsilon)^{n-N+1} x_N.$$

As $n \to \infty$, then it shows that x_n is not bounded above. As such X is not a bounded sequence. By the monotone convergence theorem (Theorem 2.9), X is not a convergent sequence. □

Example 2.43 (Bartle and Sherbert p. 70 Question 22)**.** Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence and $\{y_n\}_{n\in\mathbb{N}}$ is such that for any $\varepsilon > 0$, there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geq M$. Does it follow that $\{y_n\}_{n\in\mathbb{N}}$ is convergent?

Solution. Yes. Let $\varepsilon' = 2\varepsilon$ (which is > 0) be arbitrary. Since $\{x_n\}_{n\in\mathbb{N}}$ is convergent, then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - L| < \varepsilon$, where L is the limit of x_n. As such, choose $K = \max\{N, M\}$. Then, for all $n \geq K$, we have

$$\begin{aligned}
|y_n - L| &= |y_n - x_n + x_n - L| \\
&\leq |y_n - x_n| + |x_n - L| \quad \text{by the triangle inequality} \\
&< \varepsilon + \varepsilon \\
&= \varepsilon'
\end{aligned}$$

So, $\{y_n\}_{n\in\mathbb{N}}$ is also a convergent sequence. □

Example 2.44 (Bartle and Sherbert p. 84 Question 11)**.** Suppose

$$x_n \geq 0 \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} (-1)^n x_n \text{ exists.}$$

Show that $\{x_n\}_{n\in\mathbb{N}}$ converges.

Solution. Since the aforementioned limit, say L, exists, then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|(-1)^n x_n - L| < \varepsilon.$$

So,

$$L - \varepsilon < (-1)^n x_n < L + \varepsilon.$$

Since this inequality holds for all $n \in \mathbb{N}$, suppose n is even. Then, there exists $k \in \mathbb{N}$ such that $n = 2k$. Moreover, as $\varepsilon > 0$ can be made arbitrarily small, we can choose $\varepsilon = 1$, so

$$L - 1 < x_{2k} < L + 1.$$

On the other hand, suppose n is odd. Then, there exists $k \in \mathbb{N}$ such that $n = 2k+1$. So,

$$L - 1 < -x_{2k+1} < L + 1.$$

As $k \to \infty$, by the squeeze theorem, we see that

$$\lim_{k \to \infty} x_{2k} = L.$$

Similarly, by the squeeze theorem,

$$\lim_{k \to \infty} x_{2k+1} = -L.$$

Since $x_n \geq 0$ for all $n \in \mathbb{N}$, then $L \geq 0$ and $-L \geq 0$, which forces $L = 0$. So, $x_n \to 0$, i.e. $\{x_n\}_{n \in \mathbb{N}}$ converges. □

Example 2.45. Let $f : [a,b] \to [a,b]$ be a non-decreasing and continuous map on a closed interval such that $f(x) = x$ has no solution in (a,b). Prove that either $f(a) = a$ or $f(b) = b$. Such a point is said to be fixed under f. Furthermore, for every point $c \in (a,b)$, we may define the sequence $a_0 = c$ and $a_{n+1} = f(a_n)$. For such a sequence, prove that $\{a_n\}_{n \in \mathbb{N}}$ is monotonic and converges to that fixed point.
Hint: Draw a graph to get an idea, then provide rigorous proof. You will need the intermediate value theorem.

Solution. Define $g(x) = f(x) - x$. Since f is continuous on $[a,b]$, then the same can be said for g. We will prove the contrapositive instead. Suppose neither $f(a) = a$ nor $f(b) = b$. Then, $g(a) \neq 0$ and $g(b) \neq 0$. Also, note that $f(a) \geq a$ and $f(b) \leq b$ (since f is non-decreasing). As such, $g(a) \geq 0$ and $g(b) \leq 0$. By the intermediate value theorem, there exists $c \in (a,b)$ such that $g(c) = 0$, so $f(c) = c$, proving the first part.

For the second part, consider the case when $f(x) < x$ for all $x \in (a,b)$. In particular, $f(c) < c$, where $c \in (a,b)$. By induction, one can show that $a_{n+1} - a_n = f(a_n) - a_n < 0$, i.e. $\{a_n\}_{n \in \mathbb{N}}$ is decreasing. Since $a < c < b$, then the sequence is bounded below by a, so by the monotone convergence theorem, the sequence converges to the fixed point a. The case where $f(x) > x$ is argued similarly. □

Methods of computing square roots are numerical analysis algorithms for approximating the principal, or non-negative, square root of a real number, say S.

Theorem 2.10 (Babylonian method)**.** We start with an initial value somewhere near $\sqrt{S}$. That is $x_0 \approx \sqrt{S}$. We then use the following recurrence relation to find a better estimate for $\sqrt{S}$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{S}{x_n}\right) \quad \text{where} \quad \lim_{n \to \infty} x_n = \sqrt{S}$$

Proof. Suppose

$$\lim_{n\to\infty} x_n = L.$$

Substituting this into the recurrence relation yields

$$L = \frac{1}{2}\left(L + \frac{S}{L}\right).$$

Rearranging and the result follows. □

Theorem 2.11 (nested interval theorem)**.** Let $I_n = [a_n, b_n]$, where $n \in \mathbb{N}$, be a nested sequence of closed and bounded sequences. That is, $I_n \supseteq I_{n+1}$. Then, the intersection

$$\bigcap_{n=1}^{\infty} I_n = \{x : x \in I_n \text{ for all } n \in \mathbb{N}\}$$

is non-empty. In addition, if $b_n - a_n \to 0$ (i.e. length of I_n tends to 0), then the intersection contains exactly one point.

Definition 2.8 (harmonic numbers)**.** The harmonic numbers, H_n, are defined to be

$$\sum_{k=1}^{n} \frac{1}{k}.$$

Definition 2.9 (harmonic series)**.** The harmonic series is defined to be the following sum:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{n\to\infty} H_n$$

Note that the harmonic numbers are increasing (since $H_{n+1} - H_n > 0$) and

$$\lim_{n\to\infty} H_n = 0.$$

However, the harmonic series is divergent! Another interesting property is that other than H_1, the harmonic numbers are never integers, whose proof hinges on some elementary Number Theory.

2.3 Euler's Number, e

Definition 2.10 (Euler's number)**.** Euler's number, $e \approx 2.71828$, is defined to be

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Theorem 2.12. The sequence

$$x_n = \left(1 + \frac{1}{n}\right)^n \quad \text{is strictly increasing.}$$

That is, $x_{n+1} > x_n$ for all $n \in \mathbb{N}$.

Proof. It is easier to prove $x_n > x_{n-1}$, so we wish to prove

$$\left(1+\frac{1}{n}\right)^n > \left(1+\frac{1}{n-1}\right)^{n-1}.$$

First, we write $1+1/n$ as

$$1+\frac{1}{n-1}=\frac{n}{n-1}=\frac{1}{1-1/n}.$$

Hence,

$$\begin{aligned}\frac{(1+1/n)^n}{(1+1/(n-1))^{n-1}} &= \left(1+\frac{1}{n}\right)^n\left(1-\frac{1}{n}\right)^{n-1}\\ &= \left(1+\frac{1}{n}\right)^n\left(1-\frac{1}{n}\right)^n\left(1-\frac{1}{n}\right)^{-1}\\ &= \left(1-\frac{1}{n^2}\right)^n\left(1-\frac{1}{n}\right)^{-1}\end{aligned}$$

By Bernoulli's inequality (Theorem 1.5), this is greater than 1, and so $x_n > x_{n-1}$. □

Theorem 2.13. $2 \leq e \leq 3$

Proof. We use the series expansion of x_n.

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1+n\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\ldots+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3\\ &= 1+1+\frac{n-1}{(2!)n}+\frac{(n-1)(n-2)}{3!(n^2)}+\ldots\end{aligned}$$

It is clear that $e \geq 2$. To prove that $e \leq 3$, we consider the infinite series, but starting from the third term of the expansion of x_n. It suffices to show that

$$\frac{n-1}{2n}+\frac{(n-1)(n-2)}{6n^2}+\frac{(n-1)(n-2)(n-3)}{24n^3}+\ldots \leq 1.$$

Observe that the r^{th} term can be written as

$$\frac{(n-1)(n-2)(n-3)\ldots(n-r)}{(r+1)!n^r}=\frac{1}{(r+1)!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\ldots\left(1-\frac{r}{n}\right)\leq\frac{1}{(r+1)!}.$$

It is clear that

$$\frac{1}{(r+1)!}\leq\frac{1}{2^r},$$

since the factorial grows much faster than the geometric series, and so taking the reciprocal, the result follows. To conclude,

$$\sum_{r=1}^{\infty}\frac{(n-1)(n-2)(n-3)\ldots(n-r)}{(r+1)!n^r}\leq\sum_{r=1}^{\infty}\frac{1}{2^r}=1,$$

and we are done. □

Though the incredible constant is named after the Swiss mathematician Leonhard Euler, its discovery is actually accredited to another Swiss mathematician, Jacob Bernoulli. Just like π, e is also irrational (Theorem 2.14), which can be proven by contradiction.

Theorem 2.14. e is irrational

Proof. Suppose on the contrary that e is rational. Then, there exist $p,q\in\mathbb{Z}$ with $q\neq 0$ such that $e=p/q$. As e can be expressed as the following infinite series

$$\sum_{k=0}^{\infty}\frac{1}{k!},$$

we have

$$\begin{aligned} e=\frac{p}{q}&=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots+\frac{1}{m!}+\frac{1}{(m+1)!}+\ldots \\ m!e&=\frac{m!}{0!}+\frac{m!}{1!}+\frac{m!}{2!}+\frac{m!}{3!}+\frac{m!}{4!}+\ldots+\frac{m!}{m!}+\frac{m!}{(m+1)!}+\ldots \end{aligned}$$

By setting $q=m!$, we see that $m!e\in\mathbb{Z}$. Next, we take a look at the RHS. Observe that

$$\frac{m!}{0!}+\frac{m!}{1!}+\frac{m!}{2!}+\frac{m!}{3!}+\frac{m!}{4!}+\ldots+\frac{m!}{m!}$$

is an integer but

$$\frac{m!}{(m+1)!}+\frac{m!}{(m+2)!}+\frac{m!}{(m+3)!}+\ldots=\frac{1}{m+1}+\frac{1}{(m+1)(m+2)}+\frac{1}{(m+1)(m+2)(m+3)}+\ldots$$

is not an integer, which is a contradiction. □

2.4 The Euler-Mascheroni Constant, γ

Definition 2.11 (Euler-Mascheroni constant)**.** The Euler-Mascheroni constant, $\gamma\approx 0.5772$, is the limiting difference between the harmonic series and the natural logarithm. That is,

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right).$$

γ is an epic constant. From Figure 9, the Euler-Mascheroni constant can be regarded as the sum of areas of the yellow rectangles minus the area under the curve $y=1/x$ for $x\geq 1$.

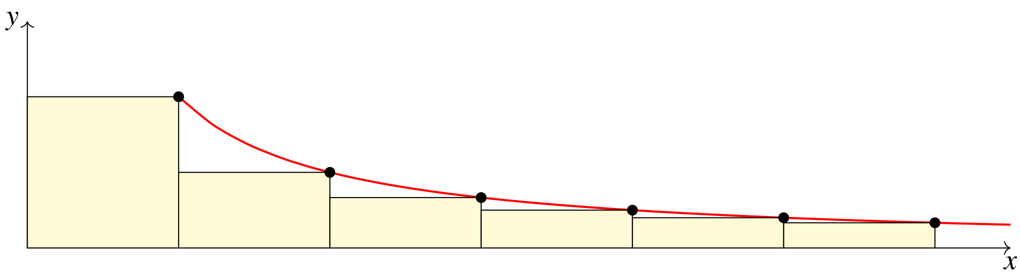


Figure 9: The graph of $y=1/x$ and an approximation for the area under the curve

It is interesting to note that the Euler-Mascheroni constant converges even though the harmonic series diverges and $\ln n$ tends to infinity as n tends to infinity. Let us prove this result using the monotone convergence theorem.

Lemma 2.1. Let x_n be the following sequence:

$$x_n=\sum_{k=1}^{n}\frac{1}{k}-\ln n$$

Then, the following properties hold:
(i) x_n is a decreasing sequence.
(ii) $0 < x_n \leq 1$, i.e. x_n is bounded.

Proof. We first prove **(i)**, i.e. $x_n > x_{n+1}$. Consider

$$x_n - x_{n+1} = \sum_{k=1}^{n} \frac{1}{k} - \ln n - \sum_{k=1}^{n+1} \frac{1}{k} + \ln(n+1) = \ln\left(\frac{n+1}{n}\right) - \frac{1}{n+1}.$$

Consider the graph of $f(x) = 1/x$, for $n \leq x \leq n+1$. We can regard

$$\begin{aligned} \ln((n+1)/n) \text{ as } &\quad \text{the area under the curve from } x=n \text{ to } x=n+1 \quad \text{and} \\ 1/(n+1) \text{ as } &\quad \text{the area of a rectangle bounded by } x=n, x=n+1 \text{ and } y=1/n \end{aligned}$$

Since f is strictly decreasing and concave up, then the area under the curve is less than the area of the rectangle. Hence, $x_n - x_{n+1} > 0$ and the result follows.

We then prove that $0 < x_n \leq 1$, i.e. x_n is bounded. Note that $x_1 = 1$. Since x_n is a strictly decreasing sequence, then

$$1 = x_1 > x_2 > x_3 > \ldots$$

and so x_n is bounded above by 1.

Write x_n as

$$\sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x}\, dx.$$

Construct a rectangle of width 1 and height $1/n$ (taking the left endpoint) and note that the sum of areas of the rectangles is strictly greater than the area under the curve, so $x_n > 0$ since the graph of f is strictly decreasing and concave up. □

With the two facts established in Lemma 2.1, by the monotone convergence theorem (Theorem 2.9), x_n converges, and it converges to γ. It is still unknown whether γ is rational or irrational. This remains an open problem.

Example 2.46 (MA2108 AY21/22 Sem 1 Midterm).
(i) Let $n \in \mathbb{N}$. Prove that

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}.$$

(ii) Use the above inequalities to prove that

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n$$

has a limit as $n \to \infty$.

Solution.
(i) Let

$$f(n) = \frac{1}{n+1}, \quad g(n) = \ln\left(1 + \frac{1}{n}\right) \quad \text{and} \quad h(n) = \frac{1}{n}.$$

Note that f, g and h are concave up on $(0, \infty)$. If we establish that $f'(n) > g'(n) > h'(n)$ for all $n \in (0, \infty)$, then we are done.

Consider
$$g'(n) - f'(n) = \frac{1}{(n+1)^2} - \frac{1}{n^2+n} = -\frac{1}{n(n+1)^2}$$
and since $n > 0$, then $g'(n) < f'(n)$.

Next, consider
$$g'(n) - h'(n) = -\frac{1}{n^2+n} + \frac{1}{n^2} = \frac{1}{n^2(n+1)}$$
and in a similar fashion, $g'(n) > h'(n)$. We are done.

(ii) It suffices to show that x_n is decreasing and bounded.

To show x_n is decreasing, consider
$$x_n - x_{n+1} = -\frac{1}{n+1} + \ln\left(1+\frac{1}{n}\right) > 0$$
by **(i)**.

To show x_n is bounded, note that $x_1 = 1$. Since x_n is a strictly decreasing sequence, then
$$1 = x_1 > x_2 > x_3 > \ldots$$
and so x_n is bounded above by 1.

Write x_n as
$$\sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x}\,dx.$$
Construct a rectangle of width 1 and height $1/n$ (taking the left endpoint) and note that the sum of areas of the rectangles is strictly greater than the area under the curve, so $x_n > 0$ since the graph of f is strictly decreasing and concave up.

Since x_n is decreasing and between 0 and 1, its limit exists. □

2.5 Subsequences

Definition 2.12 (subsequence)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, i.e. a map
$$x : \mathbb{N} \to X \quad \text{where} \quad n \mapsto x_n.$$
A subsequence of $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of the form $\{x_{n_k}\}_{k\in\mathbb{N}}$, where the indices n_k form a strictly increasing sequence of natural numbers, i.e. $n_1 < n_2 < \ldots$. Formally, it can be seen as the composition of the following two maps:
$$\mathbb{N} \xrightarrow{k} \mathbb{N} \xrightarrow{x} \mathbb{R} \quad \text{where} \quad k \mapsto n_k \mapsto x_{n_k}$$

Lemma 2.2. We have
$$\{x_n\}_{n\in\mathbb{N}} \text{ converges in } \mathbb{R} \quad \text{if and only if} \quad \text{every subsequence of } \{x_n\}_{n\in\mathbb{N}} \text{ converges in } \mathbb{R}.$$

Proof. We first prove the reverse direction. Then, $\{x_n\}_{n\in\mathbb{N}}$ as a subsequence of itself must converge in $\mathbb{R}$, i.e. consider $\mathbb{N}\to\mathbb{N}$ where $i\mapsto i$ is strictly increasing.

For the forward direction, suppose $x_n\to x$ in $\mathbb{R}$ and we have $\{x_{n_k}\}_{k\in\mathbb{N}}$ as a subsequence. Then, for all $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that for all $n\geq N$, we have $|x_n-x|<\varepsilon$. Since $k\mapsto n_k$ is strictly increasing, then for all $k\geq N$, we have $n_k\geq N$, so $|x_{n_k}-x|<\varepsilon$. □

Corollary 2.3. If $\{x_n\}_{n\in\mathbb{N}}$ has two convergent subsequences with distinct limits, then $\{x_n\}_{n\in\mathbb{N}}$ is divergent.

Theorem 2.15 (existence of monotone subsequences)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ which is monotone.

Proof. For any $k\in\mathbb{N}$, we say that

$$\{x_n\}_{n\in\mathbb{N}} \text{ has a peak at } k \quad \text{if and only if} \quad \text{for all } n\geq k \text{ we have } x_k\geq x_n.$$

Let

$$S=\left\{k\in\mathbb{N}:\{x_n\}_{n\in\mathbb{N}} \text{ has a peak at } k\right\}=\{k\in\mathbb{N}:x_k\geq x_n \text{ for all } n\geq k\}\subseteq\mathbb{N}.$$

Then, either

$$\begin{aligned}\{x_n\}_{n\in\mathbb{N}} &\text{ has infinitely many peaks} \quad \text{i.e. } S \text{ is infinite} \quad \text{or}\\ \{x_n\}_{n\in\mathbb{N}} &\text{ has finitely many peaks} \quad \text{i.e. } S \text{ is finite}\end{aligned}$$

For the first case, suppose $\{x_n\}_{n\in\mathbb{N}}$ has infinitely many peaks. Define the map $h:\mathbb{N}\to\mathbb{N}$ recursively as follows. Set $n_1=1$. Given $i\in\mathbb{N}$ such that n_i has been defined, set

$$n \quad \text{to be} \quad \text{the smallest element of the set } \left\{k\in\mathbb{N}:k>n_i \text{ and } \{x_n\}_{n\in\mathbb{N}} \text{ has a peak at } k\right\}$$

This set is $S\setminus\{1,\ldots,n_i\}$. As S is an infinite set and $\{1,\ldots,n_i\}$ is a finite set, then $S\setminus\{1,\ldots,n_i\}$ is an infinite set, which is non-empty.

By induction, for all $i\in\mathbb{N}$, we have $n_{i+1}>n_i$ in $\mathbb{N}$. So, for all $i,j\in\mathbb{N}$ such that $i<j$, we have $n_i<n_j$ in $\mathbb{N}$ and $x_{n_i}\geq x_{n_j}$ in $\mathbb{R}$. We conclude that $\{x_{n_k}\}_{k\in\mathbb{N}}$ is a monotonically decreasing subsequence of $\{x_n\}_{n\in\mathbb{N}}$.

For the second case, suppose $\{x_n\}_{n\in\mathbb{N}}$ has finitely many peaks. Then, there exists $N\in\mathbb{N}$ such that for all $k\geq N$, $\{x_n\}_{n\in\mathbb{N}}$ has a peak at k, i.e. $S\subseteq\{1,\ldots,N\}$. Define a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}\}$ recursively as follows: set $n_1=N+1$. Given n_i, choose n_{i+1} to be the smallest index $m>n_k$ such that $x_m>x_{n_k}$. Such an m always exists by the non-peak property of indices greater than N. This ensures that $\{x_{n_k}\}$ is strictly increasing. □

Theorem 2.16 (Bolzano-Weierstrass theorem)**.** Every bounded sequence has a convergent subsequence.

Example 2.47 (MA2108S AY16/17 Sem 2 Homework 4)**.** Suppose that every subsequence of x_n has a subsequence that converges to 0. Show that

$$\lim_{n\to\infty}x_n=0.^{\dagger}$$

†This also appears in MA2108 AY24/25 Sem 2 Problem Set 2 Question 32.

Solution. We first show that x_n is bounded[‡]. Suppose on the contrary that it is not. Then, for every $M,N>0$, there exists $n \geq N$ such that $|x_n| \geq M$. Then, we can find a subsequence x_{n_k} such that

$$\lim_{k\to\infty} x_{n_k} = \infty.$$

However, this subsequence does not have a subsequence that converges to 0, which is a contradiction. Hence, x_n is bounded.

Next, x_n must have only one limit point. Let L be a limit point of x_n. Then, x_{n_k} converges to L. Also, any subsequence of x_{n_k} converges to L. By the uniqueness of the limit, as every subsequence of x_{n_k} converges to 0, then x_{n_k} converges to 0, we establish the required result. □

Here is an alternative solution.

Solution. Suppose on the contrary that $\{x_n\}_{n\in\mathbb{N}}$ does not converge to 0. Then, there exists $\varepsilon>0$ such that for all $k\in\mathbb{N}$, there exists $n_k \geq k$ such that

$$|x_{n_k}| \geq \varepsilon.$$

Let $\left\{x_{n_{k_\ell}}\right\}_{\ell\in\mathbb{N}}$ be a subsequence of $\{x_{n_k}\}_{k\in\mathbb{N}}$. Then, for every $k\in\mathbb{N}$, we have $\left|x_{n_{k_\ell}}\right| \geq \varepsilon$, i.e. there exists a subsequence of $\{x_{n_k}\}_{k\in\mathbb{N}}$ that does not converge to 0, contradicting the hypothesis that $\{x_n\}_{n\in\mathbb{N}}$ does not converge to 0. □

Example 2.48 (MA2108S AY16/17 Sem 2 Homework 4)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence and for each $n\in\mathbb{N}$, let

$$s_n = \sup\{x_k : k \geq n\} \quad \text{and} \quad S = \inf\{s_n\}.$$

Show that there exists a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ that converges to S[†].

Solution. In Example 1.33, we proved that if $A\subseteq B$, where $A\neq\emptyset$, then $\sup A \leq \sup B$. From here, we claim that $\{s_n\}_{n\in\mathbb{N}}$ is monotonically decreasing. We have

$$\begin{aligned} s_{n+1}-s_n &= \sup\{x_k : k\geq n+1\} - \sup\{x_k : k \geq n\} \\ &= \sup\{x_{n+1},x_{n+2},\ldots\} - \sup\{x_n,x_{n+1},x_{n+2},\ldots\} \end{aligned}$$

If $x_n = \sup\{x_{n+1},x_{n+2},\ldots\}$, then $s_{n+1}-s_n=0$. On the other hand, if $x_n \geq \sup\{x_{n+1},x_{n+2},\ldots\}$, then $s_{n+1}-s_n \leq 0$, i.e. $\{s_n\}_{n\in\mathbb{N}}$ is decreasing. As $\{x_n\}_{n\in\mathbb{N}}$ is bounded, then $\{s_n\}_{n\in\mathbb{N}}$ is also bounded. By the monotone convergence theorem (Theorem 2.9), $\{s_n\}_{n\in\mathbb{N}}$ converges. In particular, as $\{s_n\}_{n\in\mathbb{N}}$ is bounded below and decreasing, then $\{s_n\}_{n\in\mathbb{N}}$ converges to the infimum. □

Example 2.49 (Bartle and Sherbert p. 84 Question 12)**.** Show that if $\{x_n\}_{n\in\mathbb{N}}$ is unbounded, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\frac{1}{x_{n_k}} = 0$$

Solution. Without loss of generality, suppose $\{x_n\}_{n\in\mathbb{N}}$ is not bounded above. Then, for all $M\in\mathbb{R}$, there exists $n\in\mathbb{N}$ such that for all $n\geq N$, we have $x_n > M$. Take some subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$, so for each $k\in\mathbb{N}$, choose n_k such that

$$x_{n_k} > k \quad \text{and} \quad n_1 < n_2 < \ldots < n_k.$$

Hence,

$$0 < \frac{1}{x_{n_k}} < \frac{1}{k}.$$

As $k\to\infty$, by the squeeze theorem, the result follows. □

[‡]See here for a reference.

[†]Also appears in MA2108 AY24/25 Sem 2 Problem Set 2 Question 33.

Example 2.50 (Bartle and Sherbert p. 85 Question 14)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence and let $s = \sup\{x_n : n \in \mathbb{N}\}$. Show that if $s \notin \{x_n : n \in \mathbb{N}\}$, then there is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ that converges to s.

Solution. By definition of supremum, we know that for every $\varepsilon > 0$, there exists x_n such that

$$s - \varepsilon < x_n < s \quad \text{where } s = \sup\{x_n : n \in \mathbb{N}\}.$$

Choose $\varepsilon = 1$ so

$$\text{there exists } x_{n_1} \quad \text{such that} \quad s - 1 < x_n < s.$$

Similarly, choose $\varepsilon = 1/2$ so

$$\text{there exists } x_{n_2} \quad \text{such that} \quad s - \frac{1}{2} < x_{n_2} < s.$$

As such, we obtain an increasing subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that

$$s - \frac{1}{k} < x_{n_k} < s.$$

Letting k tend to infinity, we have

$$\lim_{k\to\infty}\left(s - \frac{1}{k}\right) < \lim_{k\to\infty} x_{n_k} < \lim_{k\to\infty} s.$$

By the squeeze theorem, the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges to s. □

2.6 Cauchy Sequences

Definition 2.13 (Cauchy sequence)**.** Let F be an ordered field A sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy if

$$\text{for every } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \quad \text{such that} \quad \text{for all } m, n \geq N \text{ we have } |x_m - x_n| < \varepsilon.$$

Intuitively, what Definition 2.13 means is that for large n, the x_n's are very close to each other. For instance, the sequence $\{1/n\}_{n\in\mathbb{N}}$ is obviously Cauchy. To see why, we consider Figure 10, whereby for some $N \in \mathbb{N}$, the distance between x_m and x_n is *sufficiently small* (to be precise, this distance is at most ε, but not including it). Here, we have constructed an *open ball*[†] which contains x_n and x_m, where the distance between x_m and x_n is strictly contained in this ball!

Figure 10: If $x_n = 1/n$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy

Theorem 2.17 (convergent implies Cauchy)**.** Let F be an ordered field. If $\{x_n\}_{n\in\mathbb{N}}$ is convergent in F, then it is Cauchy.

[†]Do not need to care too much what 'open ball' means for now. Loosely speaking, you can regard it as an open interval, but this notion applies to arbitrary topological spaces.

Proof. Suppose $\{x_n\}_{in\mathbb{N}}$ is convergent in F, i.e. $x_n \to x$ in F. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|x_n - x| < \frac{\varepsilon}{2}.$$

So, for all $m, n \geq N$, we have

$$\begin{aligned}|p_n - p_m| &= |(p_n - p) - (p_m - p)| \\ &\leq |p_n - p| + |p_m - p| \quad \text{by the triangle inequality} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\end{aligned}$$

which is bounded above by ε. This shows that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. □

We take a look Examples 2.51 and 2.52 for an application of Theorem 2.17.

Example 2.51. Let $x_n = n$. Then, $\{x_n\}_{n\in\mathbb{N}}$ is not convergent, so it is not Cauchy.

Example 2.52. Let

$$y_n = \frac{1}{2^n} \quad \text{and} \quad z_n = \frac{1}{n^2}.$$

Note that $\{y_n\}_{n\in\mathbb{N}}$ is a geometric sequence with a common ratio of $1/2$ so it is convergent. As such, it is Cauchy. In fact, we can prove that $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy by directly applying Definition 2.13. Let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 1 - \frac{\ln \varepsilon}{\ln 2} \rceil$ in $\mathbb{N}$. Then, for all $m \geq n \geq N$ we have

$$|y_m - y_n| = \left| \frac{1}{2^m} - \frac{1}{2^n} \right| = \frac{|2^n - 2^m|}{2^{m+n}} \leq \frac{2^{m+1}}{2^{m+n}} = 2^{1-n} \leq 2^{1-N} < \varepsilon.$$

Moreover, $\{z_n\}_{n\in\mathbb{N}}$ is convergent, so it is Cauchy.

Remark 2.2. The converse of Theorem 2.17 is not true, i.e. if we are given a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in an ordered field F, then it may not converge to some element in F. Take for example $F = \mathbb{Q}$ and a sequence of positive rational numbers that converge to $\sqrt{2}$.

Example 2.53 (Bartle and Sherbert p. 91 Question 2)**.** Show directly from the definition that the following are Cauchy sequences:

(a) $\frac{n+1}{n}$

(b) $1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$

Solution.

(a) Let

$$x_n = \frac{n+1}{n}.$$

Then, let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 1/\varepsilon \rceil$ in $\mathbb{N}$. Then, for $m, n \in \mathbb{N}$ sufficiently large such that $m \geq n \geq N$, we have

$$|x_m - x_n| = \left| \frac{m+1}{m} - \frac{n+1}{n} \right| = \left| \frac{m-n}{mn} \right| \leq \left| \frac{m}{mn} \right| = \frac{1}{|n|} \leq \frac{1}{N} < \varepsilon.$$

(b) Let

$$x_n = 1 + \frac{1}{2!} + \ldots + \frac{1}{n!}.$$

Then, let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 1/\varepsilon \rceil$ in $\mathbb{N}$. Then, for $m, n \in \mathbb{N}$ sufficiently large such that $m, n \geq N$, we have

$$\begin{aligned}
|x_m - x_n| &= \left| \left(1 + \frac{1}{2!} + \ldots + \frac{1}{n!} + \frac{1}{(n+1)!} + \ldots + \frac{1}{m!}\right) - \left(1 + \frac{1}{2!} + \ldots + \frac{1}{n!}\right) \right| \\
&= \left| \frac{1}{(n+1)!} + \ldots + \frac{1}{m!} \right| \\
&\leq \underbrace{\frac{1}{n!} + \ldots + \frac{1}{n!}}_{m-n \text{ times}} \quad \text{since } (n+1)!, \ldots, m! \geq n! \\
&= \frac{m-n}{n!} \\
&\leq \frac{m-n}{mn} \\
&\leq \frac{m}{mn} \\
&= \frac{1}{n} < \varepsilon
\end{aligned}$$

so $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. □

Example 2.54 (Bartle and Sherbert p. 91 Question 3)**.** Show directly from the definition that the following are not Cauchy sequences:

(a) $(-1)^n$

(b) $n + \frac{(-1)^n}{n}$

(c) $\ln n$

Solution.

(a) Let $x_n = (-1)^n$. Choose $\varepsilon = 1$. Then, consider

$$|x_{n+1} - x_n| = \left|(-1)^{n+1} - (-1)^n\right| = |(-1)^n||-1-1| = 2 > 1 = \varepsilon,$$

so $\{x_n\}_{n \in \mathbb{N}}$ is not a Cauchy sequence.

(b) Let

$$x_n = n + \frac{(-1)^n}{n}.$$

Choose $\varepsilon = 1$. Then, consider

$$|x_{n+1} - x_n| = \left| n + 1 + \frac{(-1)^{n+1}}{n+1} - n - \frac{(-1)^n}{n} \right| = \left| 1 + \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^n}{n} \right|.$$

By the reverse triangle inequality,

$$|x_{n+1} - x_n| \geq 1 - \left| \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^n}{n} \right|$$

so $\{x_n\}_{n \in \mathbb{N}}$ is not a Cauchy sequence.

(c) Let $x_n = \ln n$. Choose $\varepsilon = \frac{1}{2}$. Then, consider

$$|x_{2n} - x_n| = \ln 2 > 0.5 = \varepsilon,$$

so $\{x_n\}_{n \in \mathbb{N}}$ is not a Cauchy sequence. □

Example 2.55 (Bartle and Sherbert p. 91 Question 5)**.** If $x_n = \sqrt{n}$, show that $\{x_n\}_{n \in \mathbb{N}}$ satisfies

$$\lim_{n \to \infty} |x_{n+1} - x_n| = 0 \quad \text{but that} \quad \text{it is not a Cauchy sequence.}$$

Solution. We have

$$\lim_{n\to\infty}|x_{n+1}-x_n| = \lim_{n\to\infty}\left|\sqrt{n+1}-\sqrt{n}\right| = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$$

However, we claim that $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. Consider

$$|x_{2n}-x_n| = \sqrt{2n}-\sqrt{n} = \frac{n}{\sqrt{2n}+\sqrt{n}} = \frac{\sqrt{n}}{1+\sqrt{2}}$$

which grows without bound. The result follows. □

Theorem 2.18 (Cauchy implies bounded). Let F be an ordered field. If $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in F, then it is bounded.

Proof. Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, then for all $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that for all $m,n\geq N$, we have $|x_m-x_n|<\varepsilon$. In particular, we can choose $\varepsilon=1$. Set $M=\max\{|x_1|,\ldots,|x_N|\}+1$.

Consider any $n\in\mathbb{N}$. If $n\leq N$, then we have $|x_n|\leq M$. On the other hand, if $n\geq N$, then

$$\begin{aligned}|x_n| &= |x_n-x_N+x_N|\\ &\leq |x_n-x_N|+|x_N| \quad \text{by the triangle inequality}\\ &< 1+|x_N|\leq M\end{aligned}$$

This shows that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. □

Proposition 2.3. Let F be an ordered field and $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in F. If

$$\text{there exists a subsequence of } \{x_n\}_{n\in\mathbb{N}} \text{ that converges in } F \quad \text{then} \quad \{x_n\}_{n\in\mathbb{N}} \text{ also converges in } F.$$

In fact, both limits would be the same.

Proof. Suppose $\{x_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ and $x_{n_k}\to x$ in F. Given $\varepsilon>0$, there exists $N\in N$ such that for all $k\geq N$, we have

$$|x_{n_k}-x|<\frac{\varepsilon}{2}.$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, then there exists $M\in\mathbb{N}$ such that for all $m,n\in\mathbb{N}$, we have

$$|x_n-x_m|<\frac{\varepsilon}{2}.$$

Consider the map

$$\mathbb{N}\to\mathbb{N} \quad \text{where} \quad i\mapsto n_i,$$

which is strictly increasing. As such, we can choose $k_0\geq N$ such that $n_{k_0}\geq M$, i.e. choose $k_0=\max\{N,M\}$. As such, for all $n\geq M$, we have

$$\begin{aligned}|x_n-x| &= \left|x_n-x_{n_{k_0}}+x_{n_{k_0}}-x\right|\\ &\leq \left|x_n-x_{n_{k_0}}\right|+\left|x_{n_{k_0}}-x\right| \quad \text{by the triangle inequality}\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}\end{aligned}$$

which is bounded above by ε. Then, the result follows. □

Proposition 2.4 (properties of Cauchy sequences). Let F be an ordered field. Suppose $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in F. Then, the following hold:

(i) $\{x_n + y_n\}_{n\in\mathbb{N}}$ is also Cauchy in F

(ii) $\{-x_n\}_{n\in\mathbb{N}}$ is also Cauchy in F

(iii) $\{x_n y_n\}_{n\in\mathbb{N}}$ is also Cauchy in F

We will only prove **(iii)** of Proposition 2.4 (actually, **(i)** and **(ii)** can be deduced in the midst of our proof).

Proof. Since $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ are Cauchy sequences, by Theorem 2.18, they are bounded, i.e. there exists $M > 0$ such that for all $n \in \mathbb{N}$, we have $|x_n| \leq M$ and $|y_n| \leq M$.

Also, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have

$$|x_m - x_n| < \frac{\varepsilon}{2M} \quad \text{and} \quad |y_m - y_n| < \frac{\varepsilon}{2M}.$$

As such, for every $m, n \geq N$, we have

$$\begin{aligned}
|s_n t_n - s_m t_m| &= |(s_n - s_m)t_n + s_m t_n - s_m t_m| \\
&= |(s_n - s_m)t_n + (t_n - t_m)s_m| \\
&\leq |s_n - s_m||t_n| + |t_n - t_m||s_m| \quad \text{by the triangle inequality} \\
&< \frac{\varepsilon}{2M} \cdot M + \frac{\varepsilon}{2M} \cdot M
\end{aligned}$$

which is bounded above by ε. The result follows. □

Definition 2.14 (Cauchy complete). An ordered field F is Cauchy complete if and only if every Cauchy sequence in F is convergent in F.

In general, given a sequence $\{x_n\}_{n\in\mathbb{N}}$ in an ordered field F, it is difficult to decide whether $\{x_n\}_{n\in\mathbb{N}}$ converges in F or ont. In principle, one needs to test every $x \in F$ as a possible limit. However, if F is a Cauchy complete field, then

$$\{x_n\}_{n\in\mathbb{N}} \text{ converges in } F \quad \text{if and only if} \quad \text{it is Cauchy in } F.$$

So, it suffices to just check whether $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence — a much simpler task!

Definition 2.15 (Cauchy criterion for convergence). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then, $\{x_n\}_{n\in\mathbb{N}}$ converges if and only if it is Cauchy.

Example 2.56 ($\mathbb{Q}$ is not Cauchy complete). Consider the Fibonacci sequence defined as follows:

$$\text{for all } n \in \mathbb{Z}_{\geq 0} \text{ we have } F_{n+1} = F_n + F_{n-1} \quad \text{with initial condition} \quad F_0 = F_1 = 1.$$

Let $\{x_n\}_{n\in\mathbb{Z}_{\geq 0}}$ be the sequence in $\mathbb{Q}$ defined as follows:

$$x_n = \frac{F_n}{F_{n+1}}.$$

Then, $x_0 = 1$ and

$$\text{for all } n \in \mathbb{N} \quad \text{we have} \quad x_{n+1} = \frac{1}{1+x_n} \text{ in } \mathbb{Q}.$$

This is because

$$x_{n+1} = \frac{F_{n+1}}{F_{n+2}} = \frac{F_{n+1}}{F_{n+1}+F_n} = \frac{1}{1+p_n}.$$

For all $n \in \mathbb{N}$, we claim that $1/2 \leq x_n \leq 2/3$. To see why, first note that we have $x_1 = 1/2$. By induction, if we have

$$\frac{1}{2} \leq x_n \leq \frac{2}{3} \quad \text{then} \quad \frac{3}{2} \leq 1 + x_n \leq \frac{5}{3} < 2.$$

This shows that

$$\frac{1}{2} \leq x_{n+1} = \frac{1}{1+p_n} \leq \frac{2}{3}.$$

We claim that $\{x_n\}_{n\in\mathbb{N}}$ is a contractive sequence (just to jump the gun, this appears in Definition 2.16). To see why, note that for all $n \in \mathbb{N}$, we have

$$|x_{n+1} - x_n| = \left| \frac{1}{1+x_n} + \frac{1}{1+x_{n-1}} \right| \leq \frac{|x_n - x_{n-1}|}{(1+x_n)(1+x_{n-1})} \leq \frac{4}{9} |x_n - x_{n-1}|.$$

By applying the definition of x_n recursively (this is just induction), for all $n \in \mathbb{N}$, we have

$$|x_{n+1} - x_n| \leq \frac{4}{9} |x_n - x_{n-1}| \leq \ldots \leq \left(\frac{4}{9}\right)^n |x_1 - x_0| < \left(\frac{4}{9}\right)^n.$$

Writing this compactly, we have

$$|x_{n+1} - x_n| \leq \left(\frac{4}{9}\right)^n.$$

Moreover, for all $n, r \in \mathbb{N}$, we have

$$|x_{n+r} - x_{n+r-1}| \leq \frac{4}{9} |x_{n+r-1} - x_{n+r-2}| \leq \ldots \leq \left(\frac{4}{9}\right)^{r-1} |x_{n+1} - x_n|.$$

So,

$$\begin{aligned}
|x_{n+r} - x_n| &= |x_{n+r} - x_{n+r-1} + \ldots + x_{n+1} - x_n| \\
&= |x_{n+r} - x_{n+r-1}| + \ldots + |x_{n+1} - x_n| \quad \text{by the triangle inequality} \\
&\leq \left[\left(\frac{4}{9}\right)^{r-1} + \left(\frac{4}{9}\right)^{r-2} + \ldots + 1 \right] |x_{n+1} - x_n| \\
&< \frac{1}{1 - \frac{4}{9}} |x_{n+1} - x_n| \\
&= \frac{9}{5} |x_{n+1} - x_n|
\end{aligned}$$

Again, writing this compactly, we have

$$|x_{n+r} - x_n| < \frac{9}{5} |x_{n+1} - x_n|.$$

Hence,

$$|x_{n+r} - x_n| < \frac{9}{5} \left(\frac{4}{9}\right)^n.$$

First, we claim that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$. Let $\varepsilon > 0$ be arbitrary. We shall choose $N \in \mathbb{N}$ such that $\frac{9}{5}\left(\frac{4}{9}\right)^N < \varepsilon$. Then, for all $m, n \geq N$ with $m = n + r$, we have

$$|x_m - x_n| = |x_{n+r} - x_n| < \frac{9}{5} \left(\frac{4}{9}\right)^n < \varepsilon.$$

However, $\{x_n\}_{n\in\mathbb{N}}$ does not converge in $\mathbb{Q}$. To see why, suppose on the contrary that $x_n \to x$ in $\mathbb{Q}$. Recall that for all $n \in \mathbb{N}$, we have

$$\frac{1}{2} \leq x_n \leq \frac{2}{3} \quad \text{which implies} \quad \frac{1}{2} \leq x \leq \frac{2}{3}.$$

Then, $1+x_n \to 1+x$ in $\mathbb{Q}$ and $3/2 \leq 1+x \leq 5/3$. So,

$$x_{n+1} = \frac{1}{1+x_n} \to \frac{1}{1+x} \quad \text{in } \mathbb{Q}.$$

Since $\{x_{n+1}\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ have the same limit, then

$$\frac{1}{1+x} = x \quad \text{in } \mathbb{Q}.$$

This means that $x \in \mathbb{Q}$ satisfies the equation $x^2 + x = 1$, i.e. $(2x+1)^2 = 5$ in $\mathbb{Q}$. However, there does not exist $x \in \mathbb{Q}$ such that $(2x+1)^2 = 5$.

Example 2.57 (Bartle and Sherbert p. 91 Question 7)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence such that x_n is an integer for every $n \in \mathbb{N}$. Show that $\{x_n\}_{n\in\mathbb{N}}$ is eventually constant.

Solution. Since the sequence is Cauchy, for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $m,n \geq N$, we have $|x_m - x_n| < \varepsilon$. In particular, choose $\varepsilon = \frac{1}{2}$. Since $x_n \in \mathbb{Z}$, then $x_m - x_n \in \mathbb{Z}$. The only integer with absolute value less than $\frac{1}{2}$ is 0. Therefore, for all $m,n \geq N$, $|x_m - x_n| = 0$ implies $x_m = x_n$. □

Example 2.58 (Bartle and Sherbert p. 91 Question 9)**.** If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Solution. Let $\varepsilon > 0$ be arbitrary. Let N in $\mathbb{N}$ be chosen such that

$$\frac{r^N}{1-r} < \varepsilon.$$

Then, for $m \geq n \geq N$ sufficiently large, we have

$$\begin{aligned}
|x_m - x_n| &= |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \ldots + (x_{n+1} - x_n)| \\
&\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \ldots + |x_{n+1} - x_n| \\
&\leq r^{m-1} + r^{m-2} + \ldots + r^n \\
&= \frac{r^n\left(1 - r^{m-n}\right)}{1-r} \\
&\leq \frac{r^n}{1-r} < \varepsilon
\end{aligned}$$

□

Definition 2.16 (contractive sequence)**.** Let F be an ordered field. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be contractive if

$$\text{there exists } 0 < C < 1 \quad \text{such that} \quad |x_{n+2} - x_{n+1}| \leq C\,|x_{n+1} - x_n| \text{ for all } n \in \mathbb{N}.$$

Lemma 2.3 (contractive implies convergent)**.** Every contractive sequence is convergent, and hence Cauchy.

Lemma 2.4. A sequence x_n is contractive if

$$\text{there exists } 0 < C < 1 \quad \text{such that} \quad |x_{n+2} - x_{n+1}| \leq C^{n-1}\,|x_2 - x_1| \text{ for all } n \in \mathbb{N}.$$

Proof. Repeatedly apply the inequality in Definition 2.16. □

Example 2.59 (Bartle and Sherbert p. 91 Question 11). If $y_1 < y_2$ are arbitrary real numbers and

$$y_n = \frac{1}{3}y_{n-1} + \frac{2}{3}y_{n-2} \quad \text{for } n > 2,$$

show that $\{y_n\}_{n\in\mathbb{N}}$ is convergent. What is its limit?

Solution. We have

$$\begin{aligned}
|y_n - y_{n-1}| = \left|\frac{1}{3}y_{n-1} + \frac{2}{3}y_{n-2} - y_{n-1}\right| &= \frac{2}{3}|y_{n-1} - y_{n-2}| \\
&= \left(\frac{2}{3}\right)^2 |y_{n-2} - y_{n-3}| \\
&= \ldots \quad \text{by applying the formula recursively} \\
&= \left(\frac{2}{3}\right)^{n-2} |y_2 - y_1|
\end{aligned}$$

Hence, for $m,n \in \mathbb{N}$ sufficiently large enough, where $m \geq n$, we have

$$\begin{aligned}
|y_m - y_n| &= |(y_m - y_{m-1}) + (y_{m-1} - y_{m-2}) + \ldots + (y_{n+1} - y_n)| \\
&\leq \left[\left(\frac{2}{3}\right)^{m-2} + \left(\frac{2}{3}\right)^{m-3} + \ldots + \left(\frac{2}{3}\right)^{n-1}\right] |y_2 - y_1| \\
&\leq \frac{\left(\frac{2}{3}\right)^{n-1}\left[1 - \left(\frac{2}{3}\right)^{m-n}\right]}{\frac{1}{3}} |y_2 - y_1| \\
&= 3\left(\frac{2}{3}\right)^{n-1}\left[1 - \left(\frac{2}{3}\right)^{m-n}\right] |y_2 - y_1| \\
&\leq 3\left(\frac{2}{3}\right)^{n} |y_2 - y_1|
\end{aligned}$$

This shows that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, so it converges.

We then compute the limit of the sequence. We have

$$\begin{aligned}
y_n - y_{n-1} &= -\frac{2}{3}(y_{n-1} - y_{n-2}) \\
&= \left(-\frac{2}{3}\right)^2 (y_{n-2} - y_{n-3}) \\
&= \ldots \quad \text{by applying the formula recursively} \\
&= \left(-\frac{2}{3}\right)^{n-2} (y_2 - y_1)
\end{aligned}$$

so

$$y_{n-1} - y_{n-2} = \left(-\frac{2}{3}\right)^{n-3} (y_2 - y_1)$$

and so on. Hence,

$$\begin{aligned}
(y_n - y_{n-1}) + (y_{n-1} - y_{n-2}) + \ldots + (y_3 - y_2) &= \left[\left(-\frac{2}{3}\right)^{n-2} + \left(-\frac{2}{3}\right)^{n-3} + \ldots + \left(-\frac{2}{3}\right)\right](y_2 - y_1) \\
y_n - y_2 &= \frac{\left(-\frac{2}{3}\right)\left[1 - \left(-\frac{2}{3}\right)^{n-2}\right]}{\frac{5}{3}} (y_2 - y_1)
\end{aligned}$$

Taking the limit as n goes to infinity,

$$\lim_{n\to\infty} y_n = -\frac{2}{5}(y_2 - y_1) + y_2 = \frac{2}{5}y_1 + \frac{3}{5}y_2$$

which is the desired limit of the sequence. □

Example 2.60 (MA2108 AY19/20 Sem 1). Let $a_1 \geq 0$ and for $n \geq 1$, define

$$a_{n+1} = \frac{3(1+a_n)}{3+a_n}$$

(a) Prove that a_n converges.
(b) Find the limit.

Solution.

(a) We have

$$|a_{n+2} - a_{n+1}| = \left|\frac{3(1+a_{n+1}) - 3a_{n+1} - a_{n+1}^2}{3+a_{n+1}}\right| = \left|\frac{3-a_{n+1}^2}{3+a_{n+1}}\right|,$$

which simplifies to

$$\left|\frac{3-a_n^2}{(3+a_n)(2+a_n)}\right| = \left|\frac{3-a_n^2}{3+a_n}\right| \cdot \frac{1}{|2+a_n|} < \left|\frac{3-a_n^2}{3+a_n}\right| = |a_{n+1} - a_n|$$

so a_n is a contractive sequence. By Lemma 2.3, a_n converges.

(b) Suppose

$$\lim_{n\to\infty} a_n = L.$$

Thus,

$$L = \frac{3(1+L)}{3+L}.$$

Since $L > 0$, then $L = \sqrt{3}$. □

Theorem 2.19 (equivalent characteristics of $\mathbb{R}$). Let F be an ordered field. Then, the following are equivalent:

(i) F has the least upper bound property
(ii) Every monotonically increasing sequence in F which is bounded above converges in F (precisely the monotone convergence theorem)
(iii) F is Archimedean and Cauchy complete

Proof. **(i)** implies **(ii)** follows from Proposition 2.2. We then prove **(ii)** implies **(iii)**. Suppose on the contrary that the ordered field F does not satisfy the Archimedean property. Then, there exist $a, b \in F_{>0}$ such that for all $n \in \mathbb{N}$, we have $na \leq b$, so $\{na\}_{n\in\mathbb{N}}$ is a monotonically increasing sequence in F (as $a > 0$) which is bounded above by b. As such, there exists $x \in F$ such that $na \to x$.

So, with $\varepsilon = a \in F_{>0}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|na - x| < \varepsilon = a$. We can rewrite this as $x - a < Na < x + a$, but then $x + a < (N+1)a$, which leads to a contradiction. This forces F to be Archimedean.

We then prove that F is Cauchy complete. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in F. Then, choose a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ which is monotone. Without loss of generality, assume that it is monotonically increasing. Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded above, then so is the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$. By the hypothesis, as the subsequence converges in F, then the original sequence $\{x_n\}_{n\in\mathbb{N}}$ also converges in F. Note that similar claims can be made for the case when $\{x_{n_k}\}_{k\in\mathbb{N}}$ is monotonically decreasing.

Lastly, we prove **(iii)** implies **(i)**. Suppose F is Archimedean and Cauchy complete. Let $S \subseteq F$ to be a non-empty subset which is bounded above. We wish to prove that there exists a least upper bound of S in F. Since $S \neq \emptyset$, then there exists $a_0 \in F$ such that a_0 is not the upper bound of S. To achieve this, for instance, one can choose $s_0 \in S$ and set $a_0 = s_0 - 1$. Also, since S is bounded above, then there exists $b_0 \in F$ such that b_0 is an upper bound of S. It is clear that

$$a_0 < b_0 \quad \text{in } F \quad \text{or equivalently} \quad b_0 - a_0 > 0 \quad \text{in } F.$$

Define the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ in F recursively as follows. If a_n and b_n have been defined, consider $(a_n + b_n)/2 \in F$, which would either be an upper bound of S or not.

Thereafter, set

$$a_{n+1} = \begin{cases} a_n & \text{if } \frac{a_n+b_n}{2} \text{ is an upper bound of } S; \\ \frac{a_n+b_n}{2} & \text{otherwise} \end{cases} \quad \text{and} \quad b_{n+1} = \begin{cases} \frac{a_n+b_n}{2} & \text{if } \frac{a_n+b_n}{2} \text{ is an upper bound of } S; \\ b_n & \text{otherwise} \end{cases}$$

By induction, one can deduce the following. First, a_n is not an upper bound of S but b_n is an upper bound of S. Moreover,

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n.$$

Lastly,

$$b_n - a_n = \frac{b_1 - a_1}{2^{n-1}}.$$

It is a simple exercise (one can use the formal definition of limits) to deduce that

$$\lim_{n\to\infty} (b_n - a_n) = 0.$$

We claim that $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in F. To see why, given any $\varepsilon \in F_{>0}$, since $b_n - a_n \to 0$ in F, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, one has $0 < b_n - a_n < \varepsilon$. In particular, $b_N - a_N < \varepsilon$. By using the fact that $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are Cauchy sequences, for all $m, n \in \mathbb{N}$ where $n \geq m \geq N$, we have

$$a_N \leq a_m \leq a_n \leq b_n \leq b_m \leq b_N.$$

So,

$$|a_n - a_m| \leq b_N - a_N < \varepsilon \quad \text{and} \quad |b_n - b_m| \leq b_N - a_N < \varepsilon.$$

By the Cauchy completeness of F, $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ converge in F. Let

$$a = \lim_{n\to\infty} a_n \quad \text{and} \quad b = \lim_{n\to\infty} b_n.$$

First, we claim that $a = b$ in F. We know that for all $n \in \mathbb{N}$, $a_n \leq b_n$ so $a \leq b$ in F. By way of contradiction, if $a < b$ in F, we shall define $\varepsilon = b - a > 0$ in F. By convergence (the last expression $|b_n - a_n|$ involves considering a Cauchy sequence in a Cauchy complete field, which converges), there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|a_n - a|, |b_n - b|, |b_n - a_n| \quad \text{are all } < \frac{\varepsilon}{3}.$$

Hence,

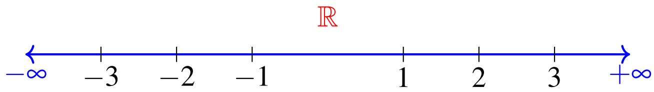
$$\varepsilon = |b - a| \leq |b - b_n| + |b_n - a_n| + |a_n - a|$$

by the triangle inequality, but the sum of terms on the right is bounded by three copies of $\varepsilon/3$, which adds to ε. As such, $\varepsilon < \varepsilon$, which is a contradiction. One can then prove that b is the least upper bound of S in F, justifying that F has the least upper bound property. □

2.7 The Extended Real Number System

We begin our discussion with the extended real numbers. Let

$$[-\infty,+\infty]=\{-\infty\}\sqcup\mathbb{R}\sqcup\{+\infty\} \quad \text{where } \sqcup \text{ denotes disjoint union.}$$



For all $x\in\mathbb{R}$, define $-\infty<x<+\infty$ which preserves the original order in $\mathbb{R}$. This is a total ordering on $[-\infty,+\infty]$. Note that for any subset E of $[-\infty,+\infty]$, we say that

$$+\infty \text{ is an upper bound of } E \quad \text{and} \quad -\infty \text{ is a lower bound of } E.$$

Thus, $\sup(E),\inf(E)$ always exist in $[-\infty,+\infty]$.

Example 2.61. We have $\sup(\emptyset)=-\infty$ and $\inf(\emptyset)=+\infty$.

Example 2.62. Let

$$A\subseteq\mathbb{R} \text{ be a set which is not bounded above in } \mathbb{R} \quad \text{and} \quad B\subseteq\mathbb{R} \text{ be a set which is not bounded below in } \mathbb{R}.$$

Then, $\sup(A)=+\infty$ and $\inf(B)=-\infty$.

Remark 2.3. The extended real number system does not form a field.

Here are some conventions. If x is real, then

$$x+\infty=\infty \quad \text{and} \quad x-\infty=-\infty \quad \text{and} \quad \frac{x}{+\infty}=\frac{x}{-\infty}=0.$$

If $x>0$, then

$$x\cdot(+\infty)=+\infty \quad \text{and} \quad x\cdot(-\infty)=-\infty.$$

Lastly, if $x<0$, then

$$x\cdot(+\infty)=-\infty \quad \text{and} \quad x\cdot(-\infty)=+\infty.$$

In contrast, in Measure Theory, the convention is that $0\cdot\{\pm\infty\}=0$. On the other hand, in Complex Analysis, $+\infty=-\infty$ in $\mathbb{C}$ but $0\cdot\infty$ is undefined.

Definition 2.17. A sequence $\{s_n\}_{n\in\mathbb{N}}$ in $[-\infty,\infty]$ converges to ∞ if and only if

$$\text{for all } A\in[-\infty,\infty] \quad \text{there exists } N\in\mathbb{N} \text{ such that for all } n\geq N \quad \text{one has } s_n>A \text{ in } [-\infty,\infty],$$

i.e. s_n is closer to ∞ than A is.

Similarly, a sequence $\{s_n\}_{n\in\mathbb{N}}$ in $[-\infty,\infty]$ converges to $-\infty$ if and only if

$$\text{for all } B\in[-\infty,\infty] \quad \text{there exists } N\in\mathbb{N} \text{ such that for all } n\geq N \quad \text{one has } s_n<B \text{ in } [-\infty,\infty],$$

i.e. s_n is closer to $-\infty$ than B is.

We write

$$\lim_{n\to\infty}s_n=\pm\infty \quad \text{or} \quad \{s_n\}_{n\in\mathbb{N}}\to\pm\infty \quad \text{in } [-\infty,\infty].$$

Proposition 2.5. Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence in $[-\infty,\infty]$ and $x \in [-\infty,\infty]$. Then, $\{s_n\}_{n\in\mathbb{N}} \to x$ in $[-\infty,\infty]$ if and only if the following properties hold:

(a) for all $A \in [-\infty,\infty]$ with $A < x$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $A < s_n$

(b) for all $B \in [-\infty,\infty]$ with $x < B$, there exists $M \in \mathbb{N}$ such that for all $n \geq M$, we have $s_n < B$

Lemma 2.5. For any sequence $\{s_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$, there exists a subsequence $\{s_{n_i}\}_{i\in\mathbb{N}}$ which converges in $[-\infty,\infty]$.

Proof. Recall Theorem 2.15 on the existence of monotone subsequences. Given a monotone subsequence $\{s_{n_i}\}_{i\in\mathbb{N}}$. If

the sequence is increasing and bounded above	then	the subsequence converges to $\mathbb{R}$
the sequence is increasing and not bounded above	then	the subsequence converges to $+\infty$
the sequence is decreasing and bounded below	then	the subsequence converges to $\mathbb{R}$
the sequence is decreasing and not bounded below	then	the subsequence converges to $-\infty$

□

2.8 Cluster Point, Limit Superior and Limit Inferior

Definition 2.18 (cluster point)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. A cluster point of a sequence is a number that is the limit of some convergent subsequence. Equivalently, a point L is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$ if every neighbourhood around L contains infinitely many terms of the sequence.

Let

$$E = \left\{\text{the limits in } [-\infty,\infty] \text{ of all convergent subsequences of } \{x_n\}_{n\in\mathbb{N}}\right\}.$$

By Lemma 2.5, E is non-empty. In fact, we call it the set of cluster points of x_n.

Definition 2.19 (limit superior and limit inferior)**.** We define the limit superior and limit inferior of x_n to be the following:

$$\limsup_{n\to\infty} x_n = \sup(E) \quad \text{and} \quad \liminf_{n\to\infty} x_n = \inf(E)$$

where E is the set of cluster points of x_n as mentioned in Definition 2.18.

Proposition 2.6. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then,

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \quad \text{in } [-\infty,\infty].$$

Equality holds if and only if $\{s_n\}_{n\in\mathbb{N}}$ converges in $[-\infty,\infty]$ in which case

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n \quad \text{in } [-\infty,\infty].$$

Proof. The set

$$E = \left\{\text{the limits in } [-\infty,\infty] \text{ of all convergent subsequences of } \{x_n\}_{n\in\mathbb{N}}\right\} \neq \emptyset$$

as mentioned earlier. Hence,

$$\liminf_{n\to\infty} x_n = \inf(E) \quad \text{is} \quad \leq \sup(E) = \limsup_{n\to\infty} x_n$$

One has

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} \quad \text{in } [-\infty,\infty]$$

if and only if either of the following hold:

(i) $E = \{x^*\}$ is a singleton subset of $[-\infty,\infty]$

(ii) there exists $x^* \in [-\infty,\infty]$ such that every convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* in $[-\infty,\infty]$

(iii) there exists $x^* \in [-\infty,\infty]$ such that every subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* in $[-\infty,\infty]$

(iv) there exists $x^* \in [-\infty,\infty]$ such that $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* in $[-\infty,\infty]$

□

Proposition 2.7 (equivalent characteristics of limit supremum). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let $x^* \in [-\infty,\infty]$. Then, the following are equivalent:

(i) We have

$$x^* = \limsup_{n\to\infty} x_n$$

(ii) For all $A \in [-\infty,\infty]$ with $A < x^*$,

$$\text{there are infinitely many } n \in \mathbb{N} \quad \text{such that } A < x_n$$

or equivalently, for all $N \in \mathbb{N}$, there exists $n \geq N$ such that $A < x_n$.

Moreover, for all $B \in [-\infty,\infty]$ with x^*,

$$\text{there are only finitely many } n \in \mathbb{N} \quad \text{such that } B \leq x_n$$

or equivalently, there exists $N \in \mathbb{N}$ usch that for all $n \geq N$, we have $x_n < B$.

(iii) We have

$$x^* = \inf\{x \in [-\infty,\infty] : \text{there are only finitely many } n \in \mathbb{N} \text{ such that } x < x_n\}$$

Example 2.63 (Bartle and Sherbert p. 85 Question 17). Alternate the terms of the sequences $\{1+\frac{1}{n}\}_{n\in\mathbb{N}}$ and $\{-\frac{1}{n}\}_{n\in\mathbb{N}}$ to obtain the sequence $\{x_n\}_{n\in\mathbb{N}}$ given by

$$2, -1, \frac{3}{2}, -\frac{1}{2}, \frac{4}{3}, -\frac{1}{3}, \frac{5}{4}, -\frac{1}{4}, \ldots.$$

Determine the values of $\limsup x_n$ and $\liminf x_n$. Also find $\sup\{x_n\}$ and $\inf\{x_n\}$.

Solution. We begin by writing the two sequences $a_n = 1+\frac{1}{n}$ and $b_n = -\frac{1}{n}$ which are interlaced to form the sequence

$$a_1, b_1, a_2, b_2, a_3, b_3, \ldots.$$

The odd-indexed subsequence is

$$x_{2k-1} = a_k = 1 + \frac{1}{k}.$$

As $k \to \infty$, we have

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right) = 1.$$

The even-indexed subsequence is

$$x_{2k} = b_k = -\frac{1}{k}.$$

As $k \to \infty$, we have

$$\lim_{k\to\infty}\left(-\frac{1}{k}\right) = 0.$$

Hence, $\limsup x_n = 1$ and $\liminf x_n = 0$.

Then, we find the supremum and infimum. The sequence $\{a_n\}_{n\in\mathbb{N}}$ is strictly decreasing and its largest term is $a_1 = 2$. Also, the even-indexed terms are all negative. Hence, $\sup\{x_n\} = 2$. The sequence $\{b_n\}_{n\in\mathbb{N}}$ is increasing (becoming less negative) with the smallest term $b_1 = -1$. The odd-indexed terms are all greater than 1. Therefore, $\inf\{x_n\} = -1$. $\square$

Example 2.64 (MA2108 AY18/19 Sem 1 Midterm). For each $n \in \mathbb{N}$, let

$$y_n = \frac{2n - \sqrt{n} + 1}{n + 2\sqrt{n} + 1}\cos\left(\frac{(n-1)\pi}{4}\right).$$

(i) Find $\limsup y_n$ and $\liminf y_n$.

(ii) Is the sequence y_n convergent? Justify your answer.

Solution.

(i) We first find $\sup y_n$. Since cosine is bounded above by 1, then

$$y_n \le \frac{2n - \sqrt{n} + 1}{n + 2\sqrt{n} + 1} = 2 - \frac{5\sqrt{n} + 1}{n + 2\sqrt{n} + 1}.$$

On the right side of the inequality, the denominator grows much faster than the numerator, so $\sup y_n = 2$. Now, we show that $\limsup y_n = 2$. Define

$$a_n = \cos\left(\frac{(n-1)\pi}{4}\right).$$

so that $a_{8n+1} = 1$ for all $n \in \mathbb{N}$. The result follows. Use the same method to find $\liminf y_n$.

(ii) No, since $\liminf y_n \ne \limsup y_n$. $\square$

Example 2.65 (MA2108 AY18/19 Sem 1 Midterm). Let a_n and b_n be bounded sequences, and let

$$c_n = \max\{a_n, b_n\} \quad \text{for all } n \in \mathbb{N}.$$

Prove that

$$\limsup c_n = \max\{\limsup a_n, \limsup b_n\}.$$

Solution. Note that $a_n, b_n \le c_n$. Define $M_1 = \limsup a_n$, $M_2 = \limsup b_n$ and $M = \max\{M_1, M_2\}$. So, $M_1 \le \limsup c_n$ and $M_2 \le \limsup c_n$. Thus, $M \le \limsup c_n$. Now, we prove that $M = \limsup c_n$.

Let c be a cluster point of c_{n_k} and $c_{n_k} \to c$. For any arbitrary $\varepsilon > 0$, there exists $K_1, K_2 \in \mathbb{N}$ such that for all $n > K_1$ and $n > K_2$, we have

$$|a_n - M_1| < \varepsilon \quad \text{and} \quad |b_n - M_2| < \varepsilon \text{ respectively.}$$

The expansion of these two inequalities yields $a_n < M_1 + \varepsilon$ and $b_n < M_2 + \varepsilon$. We'll now relate this to $c_n = \max\{a_n, b_n\}$. Let $K = \max\{K_1, K_2\}$. Then, for all $n > K$,

$$a_n < M_1 + \varepsilon < M + \varepsilon \quad \text{and} \quad b_n < M_2 + \varepsilon < M + \varepsilon.$$

Hence, $c_n < M + \varepsilon$. As mentioned, c is a cluster point of c_{n_k}, so $c_{n_k} < M + \varepsilon$. As $k \to \infty$, it is clear that $c < M + \varepsilon$. Hence, M is an upper bound for the cluster points of c_n, and so $\limsup c_n \le M$. Combining the purple inequalities yields the result. $\square$

Example 2.66 (Bartle and Sherbert p. 85 Question 19)**.** Show that if $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are bounded sequences, then

$$\limsup(x_n+y_n)\le\limsup x_n+\limsup y_n.$$

Give an example in which the two sides are not equal.

Solution. Let u be a subsequential limit of x_n+y_n. Then, there exists a subsequence $\{x_{n_k}+y_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n+y_n\}_{n\in\mathbb{N}}$ which converges to u. Let $\varepsilon>0$. Then, there exist $K_1,K_2\in\mathbb{N}$ such that

$$x_n\ge K_1 \text{ implies } x_n>\liminf x_n+\frac{\varepsilon}{2} \quad\text{and}\quad y_n\ge K_2 \text{ implies } y_n>\liminf y_n+\frac{\varepsilon}{2}.$$

Define $K=\max\{K_1,K_2\}$. Since $n_k\ge k$, then for all $k\ge K$, we have

$$u=\lim_{k\to\infty}(x_{n_k}+y_{n_k})\ge\liminf x_n+\liminf y_n+\varepsilon.$$

Since ε is some arbitrary small positive number, it follows that $\liminf x_n+\liminf y_n$ is a lower bound for x_n+y_n. As there exists a subsequence $x_{n_k}+y_{n_k}$ converging to $\inf(x_n+y_n)$, then the result follows.

For the second part, let $x_n=(-1)^n$ and $y_n=(-1)^{n+1}$. Then,

$$\limsup(x_n+y_n)=0 \quad\text{but}\quad \limsup x_n=\limsup y_n=1 \text{ so } \limsup x_n+\limsup y_n=2.$$

□

Example 2.67 (MA2108 AY19/20 Sem 1)**.** Let x_n and y_n be two bounded sequences in $\mathbb{R}$. Suppose there exists an $N\in\mathbb{N}$ such that when $n>N$, one has $x_n\le y_n$. Prove that

$$\liminf x_n\le\liminf y_n.$$

Solution. Define

$$a_n=\inf\{x_k:k\ge n\} \quad\text{and}\quad b_n=\inf\{y_k:k\ge n\}.$$

As each $x_n\le y_n$, then $\inf x_k\le\inf y_k$. That is, $a_n\le b_n$. To conclude,

$$\liminf x_n=\inf a_n\le\inf b_n=\liminf y_n.$$

□

Chapter 3
Infinite Series

3.1
Series

Let V be $\mathbb{R}$ or $\mathbb{C}$[†] We let $\{a_k\}_{k\in\mathbb{N}}$ be any sequence in V. The map

$$\sum_{k=1}^{*} a_k : \mathbb{N}\cup\{0\}\to V \quad \text{where} \quad n\mapsto\sum_{k=1}^{n} a_k \quad \text{can be defined recursively.}$$

For the case when $n=0$, we have

$$\sum_{k=1}^{0} a_k = 0_V \quad \text{where} \quad 0_V \text{ is the additive identity of } V$$

and for all $n\in\mathbb{N}\cup\{0\}$, we have

$$\sum_{k=1}^{n+1} a_k = \left(\sum_{k=1}^{n} a_k\right) + a_{n+1}.$$

This means that

$$\sum_{k=1}^{n} a_k = (\ldots((a_1+a_2)+a_3)+\ldots)+a_n.$$

From the associativity and commutativity of addition $+$ in V, one can prove that associativity and commutativity holds for n terms by induction. Hence, for all $n\in\mathbb{N}$ and permutation $\sigma\in$ set of permutations on $\{1,\ldots,n\}$, we have

$$\sum_{k=1}^{n} a_{\sigma(k)} = \sum_{k=1}^{n} a_k \quad \text{in } V.$$

Hence, given any finite set I and any map $a: I\to V$, where $i\mapsto a_i$, i.e. any finite family $\{a_i\}_{i\in I}$ of elements of V indexed by I, one can define the sum of the given series, denoted by

$$\sum_{i\in I} a_i \in V$$

as follows. First, set $n=|I|$, where $n\in\mathbb{N}\cup\{0\}$. We then choose any bijective map $\tau:\{1,\ldots,n\}\to I$. Define

$$\sum_{i\in I} a_i = \sum_{k=1}^{n} a_{\tau(k)} = a_{\tau(1)}+a_{\tau(2)}+\ldots+a_{\tau(n)}\in V.$$

This is a well-defined map which is independent of the choice of the bijection τ. With this definition, one can prove easily the following two properties in Proposition 3.1.

Proposition 3.1 (rearrangement and repartitioning). We have the following:

(i) Rearrangement: for every permutation σ on the set $\{1,\ldots,n\}$, we have

$$\sum_{i\in I} a_{\sigma(i)} = \sum_{i\in I} a_i \quad \text{in } V$$

[†]For those who are interested in MA2202, more generally, the set V can be regarded as an Abelian group, i.e. a group where the group operation is commutative.

(ii) Repartitioning: for every finite partition $\{I_j\}_{j\in J}$ of I, we have

$$\sum_{j\in J}\left(\sum_{i\in I_j} a_i\right) = \sum_{i\in I} a_i \quad \text{in } V.$$

By a finite partition, we mean that J is a finite set and for all $j \in J$, there exists $I_j \subseteq I$ such that

$$\text{for all distinct } j, j' \in J \quad \text{we have} \quad I_j \cap I_{j'} = \emptyset \text{ and } \bigcup_{j\in J} I_j = I.$$

Definition 3.1 (norm). Let V be a vector space over $\mathbb{R}$. A norm on V is a map $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ which satisfies the following properties:

(i) Positive-definite: for all $v \in V$, we have $\|v\| = 0$ if and only if $v = 0$

(ii) Homogeneity: for all $v \in V$ and $a \in \mathbb{R}$, we have $\|av\| = |a|\,\|v\|$

(iii) Triangle inequality: for all $v, w \in V$, we have $\|v+w\| \leq \|v\| + \|w\|$

A normed vector space consists of an $\mathbb{R}$-vector space V which is equipped with a norm $\|\cdot\|$ on V.

In Definition 3.1, we gave the definition of the norm of a vector. We mentioned that V is a vector space over $\mathbb{R}$, which means that the entries of V are the real numbers! Alternatively, we say that V is an $\mathbb{R}$-vector space. Note that

$$\mathbb{R} \text{ is a one-dimensional vector space over } \mathbb{R} \quad \text{but} \quad \mathbb{R} \text{ is an infinite-dimensional vector space over } \mathbb{Q}.$$

In particular, one can easily deduce that the dimension of $\mathbb{R}$ over $\mathbb{Q}$ is uncountable.

Example 3.1. $\mathbb{C}$ is a two-dimensional vector space over $\mathbb{R}$ with basis $\{1, i\}$.

Example 3.2. $\mathbb{R}^k$ and $\mathbb{C}^k$ are finite-dimensional vector spaces over $\mathbb{R}$. In $\mathbb{R}^k$, the norm function is given by the usual Euclidean k-norm, i.e.

$$\text{for any } v = (v_1, \ldots, v_k) \in \mathbb{R}^k \quad \text{we have} \quad \|v\| = \sqrt{v_1^2 + \ldots + v_k^2} \in \mathbb{R}_{\geq 0}.$$

Definition 3.2. Let V be a normed vector space and $\{a_k\}_{k\in\mathbb{N}}$ be any sequence in V. The notation

$$\sum_{k=1}^{\infty} a_k$$

is called the series in V defined by the sequence $\{a_k\}_{k\in\mathbb{N}}$. For each $n \in \mathbb{N}$, the element

$$\sum_{k=1}^{n} a_k \in V \quad \text{is} \quad \text{the } n^{\text{th}} \text{ partial sum of the series.}$$

Definition 3.3 (geometric sequence). A geometric sequence, u_n, has first term a and common ratio r. The first few terms are

$$u_1 = a,\ u_2 = ar,\ u_3 = ar^2,\ u_4 = ar^3.$$

The general term, u_n, is $u_n = ar^{n-1}$, where $n \in \mathbb{N}$.

Proposition 3.2. The sum to n terms of a geometric sequence is denoted by S_n. We establish the formula

$$S_n = \frac{a(1-r^n)}{1-r}.$$

For the sum to infinity, S_∞, we impose a restriction on r for the sum to exist. That is, $|r| < 1$. Hence, S_∞ is

$$S_\infty = \frac{a}{1-r}.$$

Remark 3.1. If $r = -1$, we obtain the famous Grandi's series $1-1+1-1+\ldots$.

Definition 3.4 (telescoping series)**.** A telescoping series is a series whose general term can be written in the form $a_n - a_{n-1}$.

Let $b_n = a_n - a_{n-1}$. Then,

$$\sum_{k=1}^{n} b_k = a_n - a_0.$$

This process is known as the method of differences. There are times when the partial fraction decomposition method has to be used on b_n (Example 3.3).

Example 3.3 (Bartle and Sherbert p. 100 Question 3)**.** By using partial fractions, show that

(a)

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1$$

(b)

$$\sum_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha} > 0 \text{ if } \alpha > 0$$

(c)

$$\sum_{n=0}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$$

Solution. These are trivial — one should recall from H2 Mathematics on how to evaluate telescoping sums using the method of difference. ☐

3.2 Properties of Convergence and Divergence

Theorem 3.1. The series

$$\sum_{k=1}^{\infty} a_k \text{ converges in } V \text{ to the sum } s \in V \quad \text{if and only if} \quad \lim_{n\to\infty} \sum_{k=1}^{n} a_k = s \quad \text{in } V.$$

Example 3.4. Suppose the series

$$\sum_{k=1}^{\infty} a_k \quad \text{has} \quad \text{only finitely many non-zero terms in } V.$$

This means that the sequence $\{a_k\}_{k\in\mathbb{N}}$ in V is eventually zero. By the formal definition of a limit, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, we have $a_k = 0_V$ in V. Then,

$$\sum_{k=1}^{\infty} a_k \quad \text{converges sequentially in } V \text{ to } \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{N} a_k \quad \text{in } V.$$

Proposition 3.3 (linearity properties of convergent series). Let

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \sum_{k=1}^{\infty} b_k \quad \text{be} \quad \text{two convergent series in } V.$$

Then, the following hold:

(i)

$$\sum_{k=1}^{\infty}(a_k+b_k) \text{ is convergent} \quad \text{and} \quad \sum_{k=1}^{\infty}(a_k+b_k) = \left(\sum_{k=1}^{\infty} a_k\right)+\left(\sum_{k=1}^{\infty} b_k\right) \quad \text{in } V$$

(ii) For every $c \in \mathbb{R}$,

$$\sum_{k=1}^{\infty} ca_k \text{ is also convergent} \quad \text{and} \quad \sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k \quad \text{in } V$$

Here is a prelude into Functional Analysis (MA4211), where we define Banach spaces (Definition 3.5).

Definition 3.5 (Banach space). A Banach space is a normed vector space V where every Cauchy sequence converges with respect to the metric induced by its norm $\|\cdot\|$.

Example 3.5. Every finite-dimensional Euclidean space is a Banach space. For example,

$$\mathbb{R}^k \text{ equipped with the norm } \|x\| = \sqrt{x_1^2+\ldots+x_n^2} \quad \text{is a Banach space.}$$

In particular, $\mathbb{R}$ and $\mathbb{C}$ are Banach spaces.

Theorem 3.2 (Cauchy criterion for series). Let V be a Banach space and $\{a_k\}_{k\in\mathbb{N}}$ be a sequence in V. The series

$$\sum_{k=1}^{\infty} a_k \text{ converges sequentially in } V \quad \text{if and only if} \quad \text{the Cauchy criterion holds.}$$

That is, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m,n \geq N$ with $m \geq n$, one has

$$\left\|\sum_{k=n+1}^{m} a_k\right\| < \varepsilon.$$

Theorem 3.3. If

$$\sum_{n=1}^{\infty} a_n \text{ converges} \quad \text{then} \quad \lim_{n\to\infty} a_n = 0.$$

The converse of Theorem 3.3 does not hold in general. That is to say, the condition $a_n \to 0$ is not sufficient to ensure the convergence of the sum of a_n. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges in } \mathbb{R}.$$

Example 3.6 (MA2108 AY18/19 Sem 1). Let

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n$$

be two series with the property that there exists $K \in \mathbb{N}$ such that

$$a_n = b_n \text{ for all } n \geq K.$$

Prove that

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \quad \text{if and only if} \quad \sum_{n=1}^{\infty} b_n \text{ is convergent.}$$

Solution. Just use Cauchy criterion. □

Example 3.7 (Bartle and Sherbert p. 270 Question 9). Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of strictly positive numbers. If

$$\sum a_n \text{ converges} \quad \text{show that } \lim_{n\to\infty} na_n = 0.$$

Give an example of a divergent series

$$\sum a_n \text{ with } \{a_n\}_{n\in\mathbb{N}} \text{ decreasing} \quad \text{for which } \lim_{n\to\infty} na_n = 0.$$

Solution. For the first part, since $\{a_n\}_{n\in\mathbb{N}}$ is decreasing and positive, for any $m \geq n$, we have

$$a_m \leq a_n \quad \text{whenever } m \geq n.$$

For $n \in \mathbb{N}$ sufficiently large, we have $a_{n+1} + a_{n+2} + \ldots + a_{2n} \geq na_{2n}$. If the sum of a_k converges, then its tail sums must go to 0. In particular,

$$\lim_{n\to\infty} (a_{n+1} + \cdots + a_{2n}) = 0.$$

Hence,

$$0 \leq \lim_{n\to\infty} na_{2n} \leq \lim_{n\to\infty} (a_{n+1} + \cdots + a_{2n}) = 0.$$

So,

$$\lim_{n\to\infty} na_{2n} = 0.$$

Finally, by monotonicity $a_n \geq a_{2n}$, and the result follows. For the second part, let

$$a_n = \frac{1}{n \ln n} \quad \text{where } n \geq 2.$$

□

Example 3.8 (Bartle and Sherbert p. 270 Question 10). Give an example of a divergent series

$$\sum a_n \text{ with } \{a_n\}_{n\in\mathbb{N}} \text{ decreasing} \quad \text{and} \quad \text{such that } \lim_{n\to\infty} na_n = 0$$

Solution. Let $a_n = \frac{1}{n \ln n}$. Then, the result follows. □

Example 3.9 (Bartle and Sherbert p. 270 Question 12). Let $a > 0$. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{1+a^n} \quad \text{is divergent if } 0 < a \leq 1 \text{ and is convergent if } a > 1.$$

Solution. We note that if $a > 1$, then $a^n > 1$, so

$$a^n < 1 + a^n < 2a^n \quad \text{which implies} \quad \frac{1}{2a^n} < \frac{1}{1+a^n} < \frac{1}{a^n}.$$

Note that

$$\sum_{n=1}^{\infty} \frac{1}{a^n} = \frac{1/a}{1-1/a} = \frac{1}{a-1} \quad \text{so} \quad \frac{1}{2(a-1)} < \sum_{n=1}^{\infty} \frac{1}{1+a^n} < \frac{1}{a-1}.$$

So, we conclude that if $a > 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{1+a^n} \text{ converges.}$$

We then claim that

$$\sum_{n=1}^{\infty} \frac{1}{1+a^n} \text{ diverges if } a = 1.$$

This is clear because

$$\sum_{n=1}^{\infty} \frac{1}{1+a^n} = \sum_{n=1}^{\infty} \frac{1}{2}$$

which is a divergent series. Lastly, we prove that the mentioned sum diverges for $0 < a < 1$. Note that

$$\lim_{n\to\infty} \frac{1}{1+a^n} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} (1+a^n)} = \frac{1}{1+0} = 1.$$

Therefore, the terms $\frac{1}{1+\alpha^n}$ do not go to 0 as $n \to \infty$. In fact, they approach 1. A necessary condition for the convergence of an infinite series (sum of a_n) is its terms $a_n \to 0$. Since $\frac{1}{1+\alpha^n}$ does not tend to 0, the series must diverge. $\square$

Example 3.10 (Bartle and Sherbert p. 277 Question 19)**.** Let $a_n > 0$ and suppose that

$$\sum a_n \text{ converges.}$$

Construct a convergent series

$$\sum b_n \text{ with } b_n > 0 \quad \text{such that} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Hence,

$$\sum b_n \text{ converges less rapidly than } \sum a_n.$$

Hint: Let A_n be the partial sums of

$$\sum a_n \quad \text{and} \quad \text{let } A \text{ denote its limit.}$$

Define $b_1 = \sqrt{A} - \sqrt{A - A_1}$ and $b_n = \sqrt{A - A_{n-1}} - \sqrt{A - A_n}$ for $n \geq 1$.

Solution. We have

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \left(\sqrt{A - A_{n-1}} - \sqrt{A - A_n}\right) = \lim_{N\to\infty} \left(\sqrt{A - A_1} - \sqrt{A - A_N}\right)$$

so

$$\begin{aligned}\sum_{n=1}^{\infty} b_n &= \sqrt{A} - \sqrt{A - A_1} + \sqrt{A - A_1} - \lim_{N\to\infty} \sqrt{A - A_N} \\ &= \sqrt{A} - \lim_{N\to\infty} \sqrt{A - A_N} \\ &= \sqrt{A}\end{aligned}$$

so the sum of b_n converges. We then prove that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

To see why this holds, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n}{\sqrt{A - A_{n-1}} - \sqrt{A - A_n}} &= \lim_{n\to\infty} \frac{a_n\left(\sqrt{A - A_{n-1}} + \sqrt{A - A_n}\right)}{A_n - A_{n-1}} \\ &= \lim_{n\to\infty} \left(\sqrt{A - A_{n-1}} + \sqrt{A - A_n}\right)\end{aligned}$$

which tends to 0 because $A_n \to A$ and $A_{n-1} \to A$. $\square$

Example 3.11 (Bartle and Sherbert p. 277 Question 20)**.** Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of real numbers converging to 0 and suppose that

$$\sum a_n \text{ diverges.}$$

Construct a divergent series

$$\sum b_n \text{ with } b_n > 0 \quad \text{such that} \quad \lim_{n\to\infty} \frac{b_n}{a_n} = 0.$$

Hence,

$$\sum b_n \text{ diverges less rapidly than } \sum a_n.$$

Hint: Let

$$b_n = \frac{a_n}{\sqrt{A_n}} \quad \text{where } A_n \text{ is the } n^{\text{th}} \text{ partial sum of } \sum a_n.$$

Solution. Since $\{a_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers converging to 0, then $a_n > 0$ for all $n \in \mathbb{N}$. So, the n^{th} partial sum of the sum of a_n is also > 0, which implies $b_n > 0$ for all $n \in \mathbb{N}$. We then note that

$$\sqrt{A_{n+1}} - \sqrt{A_n} = \frac{A_{n+1} - A_n}{\sqrt{A_{n+1}} + \sqrt{A_n}} = \frac{a_{n+1}}{\sqrt{A_{n+1}} + \sqrt{A_n}} \le \frac{a_{n+1}}{2\sqrt{A_n}} \le \frac{a_n}{2\sqrt{A_n}} = \frac{b_n}{2}.$$

By the method of difference, it follows that the sum of b_n diverges. We then prove that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0.$$

To see why this holds, we have

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{1}{\sqrt{A_n}} = 0$$

and the result follows. □

Example 3.12 (Bartle and Sherbert p. 280 Question 10)**.** If the partial sums

$$s_n \text{ of } \sum_{n=1}^{\infty} a_n \text{ are bounded} \quad \text{show that} \quad \text{the series } \sum_{n=1}^{\infty} \frac{a_n}{n} \text{ converges to } \sum_{n=1}^{\infty} \frac{s_n}{n(n+1)}.$$

Solution. Observe that $a_n = s_n - s_{n-1}$. Define T_N to be the partial sums of the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n}.$$

Then,

$$T_N = \sum_{n=1}^{N} \frac{a_n}{n} = \sum_{n=1}^{N} \frac{s_n - s_{n-1}}{n} = \sum_{n=1}^{N} \frac{s_n}{n} - \sum_{m=0}^{N-1} \frac{s_m}{m+1} = \sum_{n=1}^{N} \frac{s_n}{n} - \sum_{n=1}^{N-1} \frac{s_n}{n+1}.$$

Hence,

$$T_N = \sum_{n=1}^{N-1} \left(\frac{s_n}{n} - \frac{s_n}{n+1} \right) + \frac{s_N}{N}.$$

Since s_n is bounded, then $\frac{s_N}{N}$ tends to 0 for N sufficiently large. It follows that

$$T_N \quad \text{converges to} \quad \sum_{n=1}^{\infty} \frac{s_n}{n(n+1)}$$

□

3.3
Tests for Convergence

Theorem 3.4. A series of non-negative terms converges if and only if its partial sums form a bounded sequence.

Proof. We note that for all $k \in \mathbb{N}$, the sequence of partial sums is monotonically increasing on $\mathbb{R}$. So,

$$\begin{aligned}\text{the series converges in } \mathbb{R} \quad &\text{if and only if} \quad \text{the sequence of partial sums converges in } \mathbb{R}\\ &\text{if and only if} \quad \text{the sequence of partial sums is bounded above in } \mathbb{R}\end{aligned}$$

The result follows. □

Definition 3.6 (p-series)**.** The p-series is defined by

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Theorem 3.5 (p-series test)**.** If $p > 1$, the p-series converges. If $0 < p \leq 1$, the p-series diverges.

Example 3.13 (Bartle and Sherbert p. 101 Question 9)**.**

(a) Show that the series

$$\sum_{n=1}^{\infty} \cos n \quad \text{is not convergent.}$$

(b) Show that the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} \quad \text{is convergent.}$$

Solution.

(a) Let $N \in \mathbb{N}$. Then,

$$\begin{aligned}\sum_{n=1}^{N} \cos n \sin 1 &= \frac{1}{2} \sum_{n=1}^{N} [\sin(n+1) - \sin(n-1)]\\ &= \frac{1}{2} \sin(N+1)\\ \sum_{n=1}^{N} \cos n &= \frac{\sin(N+1)}{2 \sin 1}\end{aligned}$$

Now, it suffices to show that the limit

$$\lim_{N \to \infty} \sin N \quad \text{does not exist.}$$

Consider

$$\sin(k\pi) = 0 \quad \text{but} \quad \sin\left(\frac{\pi}{2} + 2k\pi\right) = 1$$

which shows that the aforementioned limit does not exist. Hence, the sum of $\cos n$ is not convergent.

(b) Use the fact that $-1 \leq \cos n \leq 1$, then consider the 2-series (Definition 3.6). □

Example 3.14 (Bartle and Sherbert p. 270 Question 13)**.**

(a) Does the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}} \quad \text{converge?}$$

(b) Does the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} \quad \text{converge?}$$

Solution.

(a) We have
$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\left(\sqrt{n+1}+\sqrt{n}\right)} \geq \sum_{n=1}^{\infty} \frac{1}{2(n+1)}$$
which diverges.

(b) We have
$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n\left(\sqrt{n+1}+\sqrt{n}\right)} \leq \sum_{n=1}^{\infty} \frac{1}{2n\sqrt{n}}$$
which converges. □

Theorem 3.6 (comparison test)**.** Suppose there exists $K \in \mathbb{N}$ such that $0 \leq a_n \leq b_n$ for all $n \geq K$. Then,
$$\sum_{n=1}^{\infty} b_n \text{ converges implies } \sum_{n=1}^{\infty} a_n \text{ converges} \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \text{ diverges implies } \sum_{n=1}^{\infty} b_n \text{ diverges}$$

Example 3.15 (Bartle and Sherbert p. 270 Question 8)**.** If
$$\sum a_n \text{ with } a_n > 0 \text{ is convergent} \quad \text{is } \sum a_n^2 \text{ always convergent?}$$
Either prove it or give a counterexample.

Solution. Observe that
$$\left(\sum_{n=1}^{N} a_n\right)^2 = \sum_{n=1}^{N} a_n^2 + 2\sum_{i<j} a_i a_j$$
so
$$\left(\sum_{n=1}^{N} a_n\right)^2 \geq \sum_{n=1}^{N} a_n^2.$$
Since $\sum a_n$ converges, then $\sum a_n^2$ converges too. □

Example 3.16 (Bartle and Sherbert p. 101 Question 12)**.** If
$$\sum a_n \text{ with } a_n > 0 \text{ is convergent} \quad \text{is } \sum \sqrt{a_n} \text{ always convergent?}$$
Either prove it or give a counterexample.

Solution. Not always convergent. Consider $a_n = 1/n^2$. □

Example 3.17 (Bartle and Sherbert p. 101 Question 14)**.** If
$$\sum a_n \text{ with } a_n > 0 \text{ is convergent} \quad \text{and if} \quad b_n = \frac{a_1+a_2+\cdots+a_n}{n} \text{ for } n \in \mathbb{N},$$
then show that
$$\sum b_n \text{ is not always convergent.}$$

Solution. Let $a_n = \frac{1}{2^n}$. Then, the sum of a_n converges, but
$$b_n = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{n}\left(1-\frac{1}{2^n}\right)$$
which diverges due to the presence of the harmonic series. □

Theorem 3.7 (limit comparison test)**.** Let

$$\sum_{i=n}^{\infty} a_n \text{ and } \sum_{i=n}^{\infty} b_n \quad \text{be} \quad \text{series of positive terms.}$$

Define

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

(i) If $L > 0$, then the series are either both convergent or both divergent.

(ii) If $L = 0$ and

$$\sum_{i=1}^{\infty} b_i \text{ converges} \quad \text{then} \quad \sum_{i=1}^{\infty} a_i \text{ converges.}$$

Definition 3.7 (alternating series)**.** An alternating series is a series of the form

$$\sum_{n=1}^{\infty} a_n(-1)^n = a_1 - a_2 + a_3 - a_4 + \dots \quad \text{where all } a_n > 0 \text{ or all } a_n < 0.$$

Definition 3.8 (absolute convergence)**.** Let V be a normed vector space and $\{a_n\}_{n\in\mathbb{N}}$ be any sequence in V. Then,

$$\sum_{n=1}^{\infty} a_n \text{ is absolutely convergent in } V \quad \text{if and only if} \quad \text{the series } \sum_{n=1}^{\infty} |a_n| \text{ converges.}$$

We give a classic result on the convergence of a geometric series (Theorem 3.8).

Theorem 3.8 (geometric series)**.** If $0 \le x < 1$, then

$$\sum_{n=0}^{\infty} x^n \quad \text{converges absolutely in } \mathbb{R} \text{ to } \frac{1}{1-x} \text{ in } \mathbb{R}.$$

If $x \ge 1$, the series diverges.

There is a more general result for Theorem 3.8, for which we extend it to $x \in \mathbb{C}$. More generally,

$$\begin{aligned} &\text{if } |x| < 1 \quad \text{then} \quad \sum_{n=0}^{\infty} x^n \text{ converges absolutely to } \frac{1}{1-x} \text{ in } \mathbb{C} \quad \text{and} \\ &\text{if } |x| \ge 1 \quad \text{then} \quad \sum_{n=0}^{\infty} x^n \text{ does not converge in } \mathbb{C} \end{aligned}$$

We now prove Theorem 3.8.

Proof. If $x = 1$, then for all $n \in \mathbb{N}$, the n^{th} partial sum is unbounded so the series does not converge in $\mathbb{R}$. Hence, we assume that $x \neq 1$. Recall from H2 Mathematics that for any $n \in \mathbb{N}$, the n^{th} partial sum is

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

which implies

$$\sum_{n=0}^{\infty} x^n \text{ converges in } \mathbb{R} \quad \text{if and only if} \quad x^{n+1} \text{ converges in } \mathbb{R}.$$

If $|x| < 1$, then x^{n+1} tends to 0 for large n, and

$$\sum_{n=0}^{\infty} x^n \text{ converges to } \frac{1}{1-x}.$$

The convergence is absolute because

$$\sum_{n=0}^{\infty} |x^n| = \sum_{n=0}^{\infty} |x|^n \quad \text{converges to } \frac{1}{1-|x|}.$$

On the other hand, if $|x| > 1$, then the sequence $\{x^{n+1}\}_{n\in\mathbb{N}}$ is unbounded, so it cannot converge in $\mathbb{R}$. Lastly, for the case where $|x| = 1$ but $x \neq 1$ (this can be applied to arbitrary $x \in \mathbb{C}$), we leave it as an exercise. ☐

Example 3.18 (Bartle and Sherbert p. 270 Question 11)**.** If $\{a_n\}_{n\in\mathbb{N}}$ is a sequence and if

$$\lim_{n\to\infty} n^2 a_n \quad \text{exists in } \mathbb{R},$$

show that

$$\sum a_n \quad \text{is} \quad \text{absolutely convergent.}$$

Solution. Since the aforementioned limit exists, then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left|n^2 a_n - L\right| < \varepsilon \quad \text{where} \quad \lim_{n\to\infty} n^2 a_n = L.$$

So,

$$\frac{L-\varepsilon}{n^2} < a_n < \frac{L+\varepsilon}{n^2}.$$

Taking absolute value, we have

$$|a_n| < \max\left\{\frac{L-\varepsilon}{n^2}, \frac{L+\varepsilon}{n^2}\right\} \leq \frac{|L|+\varepsilon}{n^2}.$$

Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n| \\
&\leq (N-1) \max_{i\leq 1\leq N-1} |a_i| + (|L|+\varepsilon) \sum_{n=N}^{\infty} \frac{1}{n^2} \\
&\leq (N-1) \max_{i\leq 1\leq N-1} |a_i| + (|L|+\varepsilon) \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{since } \frac{1}{n^2} \geq 0 \text{ for all } 1 \leq n \leq N-1
\end{aligned}$$

It is a well-known fact that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges — in fact to $\pi^2/6$. As such,

$$\sum_{n=1}^{\infty} |a_n|$$

is bounded above by some constant, implying that the sum of a_n is absolutely convergent. ☐

Example 3.19 (Bartle and Sherbert p. 270 Question 6)**.** Find an explicit expression for the n^{th} partial sum of

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$$

to show that this series converges to $-\ln 2$. Is this convergence absolute?

Solution. Let

$$s_N = \sum_{n=2}^{N} \ln\left(1 - \frac{1}{n^2}\right).$$

Then,

$$s_N = \sum_{n=2}^{N} \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^{N} \ln(n+1) + \ln(n-1) - 2\ln n = \ln\left(\frac{N+1}{2N}\right)$$

where the last equality uses the method of difference. Letting $N \to \infty$, we have

$$\lim_{N\to\infty} s_N = -\ln 2 + \lim_{N\to\infty} \ln\left(1 + \frac{1}{N}\right) = -\ln 2.$$

Yes, the convergence is absolute. Let

$$a_n = \ln\left(1 - \frac{1}{n^2}\right).$$

Then, for all $n \geq 2$, $a_n < 0$, so it follows that the sum of the absolute values is $\ln 2$. □

Theorem 3.9 (D'Alembert's ratio test)**.** Let

$$\sum_{i=1}^{\infty} a_n \quad \text{be} \quad \text{a series of positive terms.}$$

Define

$$L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

(i) If $L < 1$ the series converges;

(ii) if $L > 1$ the series diverges;

(iii) if $L = 1$, the test is inconclusive

Proof. We first prove **(i)**. Suppose $L < 1$. Then, one may choose $\beta \in \mathbb{R}$ such that

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < \beta < 1.$$

Then, by property of limit supremum, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, one has

$$\frac{|a_{n+1}|}{|a_n|} < \beta.$$

By induction, we see that

$$\text{for all } p \in \mathbb{N} \quad \text{we have} \quad |a_{N+p}| < \beta^p |a_N|.$$

That is to say, for all $n \geq N$, one has $|a_n| \leq \beta^{-N} |a_N| \beta^n$. Since the sum of $|a_n|$ is termwise bounded, i.e. eventually by

$$\beta^{-N} |a_N| \sum_{n=1}^{\infty} \beta^n,$$

then by the comparison test (Theorem 3.6), it follows that the sum of a_n converges absolutely.

(ii) is obvious because for all $n \geq n_0$, we have $|a_n| \geq |a_{n_0}|$. Hence, $|a_n|$ does not tend to 0, which implies that the sum of a_n cannot converge. □

We note that for **(iii)** of the ratio test (Theorem 3.9), it is possible for the sum to diverge or converge if $L = 1$. As such, it makes sense to say that when $L = 1$, the ratio test is inconclusive. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges absolutely.}$$

However, in both series, $L = 1$.

Theorem 3.10 (Cauchy's root test). We wish to determine if the series

$$\sum_{i=1}^{\infty} a_n \text{ of positive terms} \quad \text{is absolutely convergent.}$$

Define

$$L = \limsup_{n\to\infty} \sqrt[n]{a_n}.$$

(i) If $L < 1$, the series is absolutely convergent;

(ii) if $L > 1$, the series diverges;

(iii) if $L = 1$, the test is inconclusive

Proof. If $L < 1$, then one can choose

$$\beta \in \mathbb{R} \quad \text{such that} \quad \limsup_{n\to\infty} \sqrt[n]{|a_n|} < \beta < 1.$$

Then, by property of limit supremum, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, one has $\sqrt[n]{|a_n|} < \beta$. As $\beta < 1$, then

$$\sum_{n=1}^{\infty} |a_n| \quad \text{is} \quad \text{termwise dominated by } \sum_{n=1}^{\infty} \beta^n.$$

By the comparison test (Theorem 3.6), it follows that the sum of a_n converges absolutely.

On the other hand, if $L > 1$, then one can choose

$$\beta \in \mathbb{R} \quad \text{such that} \quad \limsup_{n\to\infty} \sqrt[n]{|a_n|} > \beta > 1.$$

Again, by property of limit supremum, there exist infinitely many $n \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} > \beta$. Hence, $|a_n| > 1$. As such, $|a_n|$ does not tend to 0 for large n, which implies that the sum of a_n diverges. □

We note that for **(iii)** of the root test (Theorem 3.10), it is possible for the sum of a_n to converge or diverge if $L = 1$. Hence, it makes sense to say that when $L = 1$, the root test is inconclusive. For example, we note that

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges} \quad \text{but} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges absolutely.}$$

However, in both series, $L = 1$.

Example 3.20 (Bartle and Sherbert p. 276 Question 5). Show that the series

$$\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \ldots \quad \text{is convergent}$$

but that both the ratio and the root tests fail to apply.

Solution. The sum can be written as

$$\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \ldots \leq \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which converges by the p-series test.

We then claim that the ratio test fails here. Let x_n denote the terms of the sequence. Then, $x_n = \frac{1}{n^2}$ if n is odd; $x_n = \frac{1}{n^3}$ if n is even. Note that if n is even, then

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{n^2}{(n+1)^3} < 1.$$

On the other hand, if n is odd, then

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{(n+1)^3}{n^2} > 1.$$

Hence, the ratio test does not apply here. Next, we claim that the root test also fails here. We have

$$|x_{2n}|^{1/n} = \frac{1}{(2n)^{2/n}} \quad \text{and} \quad |x_{2n+1}|^{1/n} = \frac{1}{(2n+1)^{3/n}}$$

so

$$\lim_{n\to\infty} |x_{2n}|^{1/n} = \lim_{n\to\infty} |x_{2n+1}|^{1/n} = 1 \quad \text{which implies} \quad \lim_{n\to\infty} |x_n|^{1/n} = 1.$$

Hence, the root test is inconclusive. □

At this juncture, we emphasise that the ratio test is frequently easier to apply than the root test. However, the root test has wider scope — whenever the ratio test shows convergence, the root test does too, and whenever the root test is inconclusive, the ratio test is too. That is to say, for any positive sequence of numbers $\{a_n\}_{n\in\mathbb{N}}$, we have

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \le \liminf_{n\to\infty} \sqrt[n]{a_n} \text{ in } [-\infty,\infty) \quad \text{and} \quad \limsup_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} \text{ in } (-\infty,\infty].$$

Essentially, we can combine both inequalities as well. To see why this chain of inequalities holds in the first place, it suffices to prove the one involving lim sup.

Proof. Define

$$\alpha = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

We wish to show that

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \le \alpha.$$

If $\alpha = +\infty$, then we are done. As such, we assume that $\alpha \in (-\infty,\infty)$. Let $\beta \in \mathbb{R}$ be arbitrary such that $\alpha < \beta$. Then, by property of limit supremum, there exists $N \in \mathbb{N}$ such that for all $n \ge N$, we have

$$\frac{a_{n+1}}{a_n} \le \beta.$$

For any $p \in \mathbb{N}$, we have

$$\frac{a_{N+k+1}}{a_{N+k}} \le \beta \text{ for all } 0 \le k \le p-1 \quad \text{so} \quad \frac{a_{N+p}}{a_N} \le \beta^p.$$

Here, we have used the cancellation property of a telescoping product. That is to say, for all $n \ge N$, one has $a_n \le \beta^{-N} a_N \beta^n$, where we set $n = N+p$, i.e. $\sqrt[n]{a_n} \le \sqrt[n]{a_N \beta^{-N}} \cdot \beta$. Hence,

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \le \beta$$

and the result follows. □

We take a look at Example 3.21 which discusses the superiority of the root test in comparison to the ratio test.

Example 3.21 (root test stronger than ratio test)**.** Let

$$a_1 = \frac{1}{2} \quad a_2 = \frac{1}{3} \quad a_3 = \frac{1}{2^2} \quad a_4 = \frac{1}{3^2},$$

and I believe that you get the idea from here. Then,

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \dots.$$

Here, we see that

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}} < 1 \quad \text{but} \quad \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2}\right)^n = +\infty.$$

This shows that the root test indicates convergence but the ratio test fails!

Example 3.22 (Bartle and Sherbert p. 270 Question 7)**.**

(a) If

$$\sum a_n \text{ is absolutely convergent} \quad \text{and} \quad \{b_n\}_{n\in\mathbb{N}} \text{ is a bounded sequence,}$$

show that

$$\sum a_n b_n \text{ is absolutely convergent}$$

(b) Give an example to show that if the convergence of

$$\sum a_n \text{ is conditional and } \{b_n\}_{n\in\mathbb{N}} \text{ is a bounded sequence} \quad \text{then} \quad \sum a_n b_n \text{ may diverge}$$

Solution.

(a) Since

$$\sum a_n \text{ is absolutely convergent,}$$

then suppose the limit of the sum of absolute values is L_1, i.e. for every $\varepsilon_1 > 0$, there exists $N \in \mathbb{N}$ such that

$$||a_1| + \ldots + |a_n| - L| < \varepsilon.$$

Since $\{b_n\}_{n\in\mathbb{N}}$ is bounded, then there exists $M \in \mathbb{R}$ such that $-M \le b_n \le M$ for all $n \in \mathbb{N}$. Hence, for n sufficiently large, we have

$$-M(|a_1| + \ldots + |a_n|) \le |a_1b_1| + |a_2b_2| + \ldots + |a_nb_n| \le M(|a_1| + \ldots + |a_n|)$$

Since

$$L - \varepsilon < |a_1| + \ldots + |a_n| < L + \varepsilon,$$

then

$$-ML + \varepsilon M < |a_1b_1| + \ldots + |a_nb_n| < ML + \varepsilon M$$

so

$$||a_1b_1| + |a_nb_n| - \varepsilon M| < ML$$

so the sum a_nb_n is absolutely convergent.

(b) Let $a_n = \frac{(-1)^n}{n}$ and $b_n = (-1)^n$, then the sum of a_nb_n is the harmonic series, which diverges! □

Theorem 3.11 (Cauchy's condensation test)**.** For a non-increasing sequence of non-negative real numbers $f(n)$,

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \quad \text{if and only if} \quad \text{the condensed series } \sum_{n=0}^{\infty} 2^n f(2^n) \text{ converges.}$$

Observe the difference in the lower indices — one of them is 1 and another is 0.

Proof. Suppose the original series converges. We wish to prove that the condensed series converges. Consider twice the original series.

$$\begin{aligned}2\sum_{n=1}^{\infty} f(n) &= (f(1)+f(1)) + (f(2)+f(2)+f(3)+f(3)) + \dots \\ &\geq (f(1)+f(2)) + (f(2)+f(4)+f(4)+f(4)) + \dots \\ &= f(1) + (f(2)+f(2)) + (f(4)+f(4)+f(4)+f(4)) + \dots \\ &= \sum_{n=0}^{\infty} 2^n f(2^n)\end{aligned}$$

Dividing both sides by 2, the condensed series converges.

Now, suppose the condensed series converges. We wish to prove the original series converges.

$$\begin{aligned}\sum_{n=0}^{\infty} 2^n f(2^n) &= f(1)+f(2)+f(2)+f(4)+f(4)+f(4)+f(4)+\dots \\ &\geq f(1)+f(2)+f(3)+f(4)+f(5)+f(6)+f(7)+f(8)+\dots \\ &= \sum_{n=1}^{\infty} f(n)\end{aligned}$$

This concludes the proof. □

Corollary 3.1. If both series converge, the sum of the condensed series is no more than twice as large as the sum of the original. We have the inequality

$$\sum_{n=1}^{\infty} f(n) \leq \sum_{n=0}^{\infty} 2^n f(2^n) \leq 2\sum_{n=1}^{\infty} f(n).$$

Corollary 3.2. Consider a variant of the p-series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}.$$

If $p > 1$, the series converges. If $p \leq 1$, the series diverges.

Proof. We use Cauchy's condensation test (Theorem 3.11). Note that

$$f(n) = \frac{1}{n(\ln n)^p},$$

so

$$2^n f(2^n) = \frac{2^n}{2^n(\ln(2^n))^p} = \frac{1}{n^p(\ln 2)^p}.$$

We have

$$\frac{1}{(\ln 2)^p}\sum_{n=2}^{\infty}\frac{1}{n^p}$$

so the result follows by the conventional p-series test. □

Theorem 3.12 (partial summation formula)**.** Given two sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ indxed by the non-negative integers $\mathbb{Z}_{\geq 0}$, set $A_{-1} = 0$ and for any $n \geq 0$, put

$$A_n = \sum_{k=0}^{n} a_k.$$

Then, for any $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \left(\sum_{n=p}^{q-1} A_n (b_n - b_{n+1})\right) + A_q b_q - A_{p-1} b_p.$$

Proof. This is easy to see because

$$\begin{aligned}\sum_{n=p}^{q} a_n b_n &= \sum_{n=p}^{q} (A_n - A_{n-1}) b_n \\ &= \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n \\ &= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}\end{aligned}$$

and the result follows from here. □

We note that the partial summation formula (Theorem 3.12) is useful in the investigation of series of the form

$$\sum_{n=1}^{\infty} a_n b_n.$$

Theorem 3.13 (Dirichlet's test)**.** Suppose the partial sums A_n form a bounded sequence in $\mathbb{R}$, $\{b_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of numbers such that

$$\lim_{n\to\infty} b_n = 0.$$

Then,

$$\sum_{n=0}^{\infty} a_n b_n \quad \text{converges in } \mathbb{R}.$$

Proof. Since $\{A_n\}_{n\in\mathbb{N}}$ forms a bounded sequence, then there exists $M \geq 0$ such that for all $n \in \mathbb{Z}_{\geq 0}$, we have $|A_n| \leq M$. Let $\varepsilon > 0$ be arbitrary. As

$$\lim_{n\to\infty} b_n = 0,$$

then there exists $N \in \mathbb{Z}_{\geq 0}$ such that for all $n \geq N$, we have $0 \leq b_n \leq \frac{\varepsilon}{2M}$. As such, for all $p, q \geq N$ with $p \leq q$, we apply the partial summation formula (Theorem 3.12) to obtain

$$\begin{aligned}\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\ &\leq \sum_{n=p}^{q-1} |A_n| |b_n - b_{n+1}| + |A_q| |b_q| + |A_{p-1}| |b_p| \quad \text{by the triangle inequality} \\ &\leq M \left(\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p\right) \quad \text{since } \{b_n\}_{n\in\mathbb{N}} \text{ is a decreasing sequence} \\ &\leq 2M b_p\end{aligned}$$

Since $2Mb_p \leq \varepsilon$, then it follows that the partial sums of the sum of $a_n b_n$ form a Cauchy sequence in $\mathbb{R}$. As such, the aforementioned sum converges in $\mathbb{R}$. $\square$

Example 3.23 (Fourier series). For example,

$$\sum_{n=1}^{\infty} \frac{\cos n}{\sqrt{n}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n}} \quad \text{converge.}$$

More generally, for any sequence of numbers $\{b_n\}_{n\in\mathbb{N}}$ which decreases to 0 in $\mathbb{R}$, and for any $x \in \mathbb{R} \setminus 2\pi\mathbb{Z}$,

$$\sum_{n=1}^{\infty} b_n \cos nx \quad \text{and} \quad \sum_{n=1}^{\infty} b_n \sin nx \quad \text{converge in } \mathbb{C}.$$

These are known as a Fourier cosine series and a Fourier sine series respectively. In many contexts, a function can be represented as a sum of sine and cosine series, which together form a full Fourier series. Fourier cosine series are typically used for even extensions of functions, while Fourier sine series are used for odd extensions.

By Dirichlet's test (Theorem 3.13), it suffices to show that the partial sums of the Fourier cosine and Fourier sine series are bounded, i.e. we should show something like

$$\sum_{n=1}^{N} \cos nx \text{ and } \sum_{n=1}^{N} \sin nx \quad \text{are} \quad \text{bounded.}$$

Indeed, by Lagrange's trigonometric identities (can be easily proved using techniques taught in H2 Mathematics), we have

$$\left|\sum_{k=1}^{n} \cos kx\right| = \left|\frac{\sin\left(\left(n+\frac{1}{2}\right)x\right) - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}\right| \leq \frac{1}{\left|\sin\frac{1}{2}x\right|} \quad \text{and} \quad \left|\sum_{k=1}^{n} \sin kx\right| = \left|\frac{\cos\frac{1}{2}x - \cos\left(\left(n+\frac{1}{2}x\right)\right)}{2\sin\frac{1}{2}x}\right| \leq \frac{1}{\left|\sin\frac{1}{2}x\right|}$$

which hold for all $x \in \mathbb{R} \setminus 2\pi\mathbb{Z}$.

Example 3.24 (Bartle and Sherbert p. 280 Question 6). Let $\{a_n\}_{n\in\mathbb{N}}$ be a real sequence and let $p < q$. If

$$\sum_{n=1}^{\infty} \frac{a_n}{n^p} \text{ converges} \quad \text{show that} \quad \text{the series } \sum_{n=1}^{\infty} \frac{a_n}{n^q} \text{ also converges.}$$

Solution. We observe that

$$\frac{a_n}{n^q} = \frac{a_n}{n^p} \cdot \frac{1}{n^{q-p}}.$$

Since the sum of $\frac{a_n}{n^p}$ converges, then it is bounded above by some constant M, and that the sequence formed by $\frac{1}{n^{q-p}}$ is decreasing and tends to 0, by Dirichlet's test (Theorem 3.13), the result follows. $\square$

Example 3.25 (Bartle and Sherbert p. 280 Question 9). If the partial sums of

$$\sum_{n=1}^{\infty} a_n \text{ are bounded} \quad \text{show that} \quad \text{the series } \sum_{n=1}^{\infty} a_n e^{-nt} \text{ converges for } t > 0.$$

Solution. Since $\{e^{-nt}\}_{n\in\mathbb{N}}$ is a decreasing sequence of numbers which tends to 0, by Dirichlet's test (Theorem 3.13, the result follows. $\square$

We then discuss the alternating series test (Theorem 3.14), which can be seen as a special case of Dirichlet's test (Theorem 3.13).

Theorem 3.14 (alternating series test). If a_n is an alternating series with

$$\left|\frac{a_{n+1}}{a_n}\right| \leq 1 \quad \text{for } n \geq 1, \text{ i.e. } a_n \text{ decreases monotonically} \quad \text{and} \quad \lim_{n\to\infty} a_n = 0,$$

then the sum of a_n converges.

Example 3.26 (alternating harmonic series). The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \quad \text{converges} \quad \text{by the alternating series test.}$$

Example 3.27 (MA2108 AY21/22 Sem 1 Midterm). Consider the following alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2/3}}.$$

Is it convergent? Prove your conclusion.

Solution. Let

$$a_n = \frac{1}{n^{2/3}} = n^{-2/3}.$$

We verify if

$$\lim_{n\to\infty} a_n = 0$$

and a_n is monotonically decreasing. The limit property is obviously true.

Consider

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{-2/3}}{n^{-2/3}}\right| = \left|\left(1+\frac{1}{n}\right)^{-2/3}\right| < 1$$

and so $a_{n+1} < a_n$. By the alternating series test (Theorem 3.14), the series is convergent. □

Example 3.28 (Bartle and Sherbert p. 280 Question 1). For each of the following series, determine whether it converges absolutely and whether it converges (sequentially):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n+2}$ **(d)** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$

Solution.

(a) The series converges absolutely by the p-series test. Hence, the original series converges.

(b) The series converges but it does not converge absolutely. For the latter, it can be easily justified by comparing it to the harmonic series.

(c) The series converges but it does not converge absolutely. For the latter, it can be easily justified by comparing it to the harmonic series.

(d) The series converges but not absolutely. The former is a simple application of the alternating series test (Theorem 3.14). For the latter, we note that for $n > e$, we have $\ln n > 1$ so by comparing

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

with the harmonic series. □

Example 3.29 (Bartle and Sherbert p. 280 Question 5)**.** Consider the series

$$1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\ldots,$$

where the terms come in pairs of signs. Does it converge?

Solution. Let

$$a_n=\frac{1}{2n}+\frac{1}{2n+1}.$$

Then, the sum can be written as

$$1-a_1+a_2-a_3+a_4+\ldots=1+\sum_{n=1}^{\infty}(-1)^n a_n=1+\sum_{n=1}^{\infty}\frac{(-1)^n(4n+1)}{2n(2n+1)}.$$

Define

$$b_n=\frac{4n+1}{2n(2n+1)}.$$

Then,

$$b_{n+1}-b_n=\frac{4n+5}{(2n+2)(2n+3)}-\frac{4n+1}{2n(2n+1)}=-\frac{6n+7}{(2n+1)(2n+2)(2n+3)}$$

which shows that $\{b_n\}_{n\in\mathbb{N}}$ is a decreasing sequence. In fact, b_n tends to 0. By the alternating series test (Theorem 3.14), we conclude that the original series converges. □

3.4
Grouping and Rearrangement of Series

Theorem 3.15 (convergence is stable under grouping)**.** Let V be a Banach space. If

$$\sum_{n=1}^{\infty}a_n \quad \text{converges absolutely in } V,$$

then any series obtained by

$$\text{grouping the terms of } \sum_{n=1}^{\infty}a_n \text{ is also absolutely convergent in } V \quad \text{and} \quad \text{has the same value as } \sum_{n=1}^{\infty}a_n.$$

Definition 3.9 (rearrangement)**.** A series

$$\sum_{n=1}^{\infty}b_n \quad \text{is} \quad \text{a rearrangement of the series } \sum_{n=1}^{\infty}a_n$$

if there is a bijection $f:\mathbb{N}\to\mathbb{N}$ such that $b_n=a_{f(n)}$ for all $n\in\mathbb{N}$.

Theorem 3.16 (absolute convergence is stable under rearrangement)**.** Let V be a Banach space. Suppose the series

$$\sum_{k=1}^{\infty}a_k \quad \text{converges absolutely in } V.$$

Then, for all $\sigma\in$ set of permutations of $\mathbb{N}$, the series

$$\sum_{k=1}^{\infty}a_{\sigma(k)} \text{ also converges absolutely in } V \quad \text{and} \quad \sum_{k=1}^{\infty}a_{\sigma(k)}=\sum_{k=1}^{\infty}a_k \text{ in } V.$$

We can generalise to the following result. Given a Banach space V, for any countably infinite set I and any map $a : I \to V$, where $k \mapsto a_k$, the series

$$\sum_{i\in I} a_i \quad \text{is} \quad \text{absolutely convergent in } V$$

if and only if for every bijection $\tau : \mathbb{N} \to I$, the series

$$\sum_{k=1}^{\infty} a_{\tau(k)} \quad \text{converges absolutely in } V \quad \text{or equivalently} \quad \sum_{k=1}^{\infty} \left\| a_{\tau(k)} \right\| < \infty.$$

When this happens, we define

$$\sum_{i\in I} a_i = \sum_{k=1}^{\infty} a_{\tau(k)} \in V \quad \text{which is} \quad \text{called the sum of the given series.}$$

In fact, this is well-defined and independent of the choice of the bijection τ. Hence, for every $\varepsilon > 0$, there exists a finite set $I_0 \subseteq I$ such that for every finite set $I' \subseteq I$ with $I_0 \subseteq I'$, we have

$$\left\| \sum_{i\in I'} a_i - \sum_{i\in I} a_i \right\| < \varepsilon.$$

It follows that we have the triangle inequality for absolutely convergent series, which states that

$$\left\| \sum_{i\in I} a_i \right\| \leq \sum_{i\in I} \|a_i\| \quad \text{in } \mathbb{R}_{\geq 0}.$$

Corollary 3.3 (rearrangement). If

$$\sum_{i\in I} a_i \quad \text{is} \quad \text{absolutely convergent in } V,$$

then for every permutation $\sigma \in$ the set of permutations of I,

$$\sum_{i\in I} a_{\sigma(i)} \text{ is absolutely convergent in } V \quad \text{and} \quad \sum_{i\in I} a_{\sigma(i)} = \sum_{i\in I} a_i \text{ in } V.$$

The proof of Corollary 3.3 is immediate from the well-defined property of the sum of a_i, where $i \in I$.

Corollary 3.4 (repartitioning). Let V be a Banach space. If

$$\sum_{i\in I} a_i \quad \text{is} \quad \text{absolutely convergent in } V,$$

then for every partition $\{I_j\}_{j\in J}$ of I, we have the following:

(i) for all $j \in J$, the series

$$\sum_{i\in I_j} a_i \quad \text{converges absolutely in } V$$

(ii) the series

$$\sum_{j\in J} \left(\sum_{i\in I_j} a_i \right) \quad \text{converges absolutely in } V$$

(iii) we have

$$\sum_{j\in J} \left(\sum_{i\in I_j} a_i \right) = \sum_{i\in I} a_i \quad \text{in } V$$

Example 3.30 (paradoxical?)**.** The series

$$1+2+3+4+\dots$$

is an interesting one. Although it is a divergent series, by certain methods such as rearrangement of the original series or by Ramanujan summation, we obtain the formula

$$1+2+3+4+\dots = -\frac{1}{12}.$$

Example 3.31 (MA2108S AY16/17 Sem 2 Homework 4)**.** For x_n given by the following formulae, establish either the convergence or the divergence of the series

$$\sum_{n=1}^{\infty} x_n.$$

(a) $x_n = \dfrac{n}{n+1}$ **(b)** $x_n = \dfrac{(-1)^n n}{n+1}$ **(c)** $x_n = \dfrac{n^2}{n+1}$ **(d)** $x_n = \dfrac{2n^2+3}{n^2+1}$

Solution.

(a) Note that

$$x_n = 1 - \frac{1}{n+1}$$

so

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty}\left(1-\frac{1}{n+1}\right)$$

which diverges.

(b) Pairing the terms,

$$\begin{aligned}\sum_{n=1}^{\infty} x_n &= (x_1+x_2)+(x_3+x_4)+(x_5+x_6)+\dots \\ &= \frac{1}{6}+\frac{1}{20}+\frac{1}{42}+\frac{1}{72}+\dots \\ &= \sum_{n=1}^{\infty}\frac{1}{(2n)(2n+1)} \\ &= \frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}+\dots\end{aligned}$$

The above alternating sum motivates us to use the infinite series representation of $\ln 2$, so the sum of x_n is $1-\ln 2$, implying that x_n converges.

(c) x_n can be written as

$$x_n = n-1+\frac{1}{n+1}$$

so the sum is divergent.

(d) x_n can be written as

$$x_n = 2+\frac{1}{n^2+1}$$

so the sum is divergent. □

Example 3.32 (MA2108 AY18/19 Sem 1)**.** Determine whether each of the following sequences is convergent. Justify your answers.

(i)

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}+\sqrt{n^2+1}}$$

(ii)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{2n^2-\cos n}$$

(iii)

$$\sum_{n=1}^{\infty} x_n,$$

where x_n is defined to be the following:

$$x_n = \frac{3^{n+1}}{(n+1)!} \text{ if } n \text{ is odd}, \; x_n = -\frac{3^{n-1}}{(n-1)!} \text{ if } n \text{ is even.}$$

Solution.

(i) We use the alternating series test as $(-1)^{n+1}$ is present here. Define

$$a_n = \frac{1}{\sqrt{n}+\sqrt{n^2+1}}.$$

We prove that a_n is monotonically decreasing. Note that

$$a_{n+1} = \frac{1}{\sqrt{n+1}+\sqrt{(n+1)^2+1}}.$$

It is clear that $a_n > a_{n+1}$ because $\sqrt{n} < \sqrt{n+1}$ and $\sqrt{n^2+1} < \sqrt{(n+1)^2+1}$, thus the sequence is decreasing. Lastly,

$$\lim_{n\to\infty} a_n = 0 \quad \text{so} \quad \text{the series converges by the alternating series test.}$$

(ii) We use the limit comparison test. Let

$$a_n = \frac{\sqrt{n}+1}{2n^2-\cos n} \text{ and } b_n = \frac{1}{n^{3/2}}.$$

Note that b_n is the p-series, where $p = 3/2$, so b_n converges. Consider

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+n^{3/2}}{2n^2-\cos n} = \lim_{n\to\infty} \frac{1+n^{-1/2}}{2-\cos n/n^2} = \frac{1}{2}.$$

As this limit is finite, the series converges by the limit comparison test.

(iii) The sum of x_n is a rearrangement of the following alternating series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n!}$$

Use the ratio test to prove that this alternating series converges. □

Example 3.33 (Bartle and Sherbert p. 276 Question 1). Establish the convergence or the divergence of the series whose n^{th} term is

(a) $\frac{1}{(n+1)(n+2)}$ **(b)** $\frac{n}{(n+1)(n+2)}$ **(c)** $2^{-1/n}$ **(d)** $\frac{n}{2^n}$

Solution.

(a) Converges — use method of difference.

(b) Diverges — use method of difference.

(c) Diverges. Use the fact that

$$\sum_{n=1}^{\infty} \frac{1}{2^{1/n}} \geq \sum_{n=1}^{\infty} \frac{1}{2}.$$

(d) Converges[†]. The trick is to first let the sum be S. Then,

$$\begin{aligned} S &= \frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\frac{5}{2^5}+\ldots \\ &= \left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\ldots\right)+\left(\frac{1}{2^2}+\frac{2}{2^3}+\frac{3}{2^4}+\frac{4}{2^5}+\ldots\right) \\ &= \left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\ldots\right)+\frac{1}{2}S \\ &= 1+\frac{1}{2}S \end{aligned}$$

So, $S=2$.

Here is an interesting perspective to the arithmetic-geometric series in **(d)** of Example 3.33. Let X be a random variable denoting the number of occurrences up to and including the first occurrence of a heads. Then,

$$P(X=k)=\frac{1}{2}\cdot\left(\frac{1}{2}\right)^{k-1}=\frac{1}{2^k}$$

as this is equivalent to saying that the first $k-1$ trials are failures (i.e. tails) and the k^{th} trial is a success (i.e. a head). This essentially models a geometric distribution with probability of success $1/2$. So, we write $X\sim\text{Geo}(1/2)$. One notes that the expectation can be computed as follows:

$$\begin{aligned} E(X) &= 1\cdot P(X=1)+2\cdot P(X=2)+3\cdot P(X=3) \\ &= 1\cdot\frac{1}{2}+2\cdot\frac{1}{2^2}+3\cdot\frac{1}{2^3}+\ldots \end{aligned}$$

It is a well-known fact that the expectation can be computed easily — if $X\sim\text{Geo}(p)$, then $E(X)=1/p$. Since $p=1/2$, then the expectation is 2.

Example 3.34 (Bartle and Sherbert p. 276 Question 2)**.** Establish the convergence or divergence of the series whose n^{th} term is:

(a) $\dfrac{1}{\sqrt{n(n+1)}}$ **(b)** $\dfrac{1}{\sqrt{n^2(n+1)}}$ **(c)** $\dfrac{n!}{n^n}$ **(d)** $\dfrac{(-1)^n n}{n+1}$

Solution.

(a) Diverges. Use the fact that

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}\geq\sum_{n=1}^{\infty}\frac{1}{\sqrt{(n+1)^2}}=\sum_{n=1}^{\infty}\frac{1}{n+1}.$$

(b) Converges. Use the fact that

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2(n+1)}}\leq\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3}}.$$

(c) Converges. By the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right|=\lim_{n\to\infty}\left|(n+1)\cdot\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n+1}\right|=\frac{1}{e}<1.$$

(d) Diverges by the alternating series test. □

[†]This is known as an arithmetic-geometric series since it is the product of an arithmetic sequence and a geometric sequence. In fact, it is known as Gabriel's staircase.

Theorem 3.17 (Riemann rearrangement theorem). If a series

$$\sum_{n=1}^{\infty} a_n$$

of real numbers converges conditionally (i.e. it converges, but the series of absolute values diverges), then for any real number L there exists a rearrangement of the terms of the series such that the rearranged series converges to L. Moreover, it is also possible to rearrange the series so that it diverges to $+\infty$ or $-\infty$, or even fails to have a limit in the extended real sense.

Example 3.35 (alternating harmonic series). Recall that the alternating harmonic series is given by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots,$$

and it is well-known that this series converges to $\ln 2$, but it does not converge absolutely. The Riemann rearrangement theorem (Theorem 3.17) tells us that for any real number L, there exists a rearrangement of the terms of a conditionally convergent series (like the alternating harmonic series) that converges to L. The idea behind the rearrangement is as follows.

We first accumulate the positive terms. The positive terms are

$$P = \left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\right\}.$$

Add these terms until the partial sum exceeds the target L. We then consider the negative terms

$$N = \left\{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, -\frac{1}{8}, \ldots\right\}.$$

Add enough negative terms to bring the partial sum below L. Continue alternating between adding positive terms until the sum exceeds L and then negative terms until it drops below L. This process creates a sequence of partial sums that oscillate around L with the oscillations diminishing in size, ensuring convergence to L in the limit.

Chapter 4
Continuity

4.1
Metric Spaces

Definition 4.1 (metric). A metric on a set X is a map $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfying the following properties:

(i) Positive-definiteness: for all $x_1, x_2 \in X$, we have $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$

(ii) Symmetry: for all $x_1, x_2 \in X$, we have $d(x_1, x_2) = d(x_2, x_1)$

(iii) Triangle inequality: for all x_1, x_2, x_3, we have $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$

A metric space consists of a set X together with a metric on d on X.

Proposition 4.1. For a normed vector space V and any subset $X \subseteq V$,

$$\text{the map } d : X \times X \to \mathbb{R} \text{ where } d(X_1, x_2) = \|x_1 - x_2\|_V \quad \text{is} \quad \text{a metric on } X.$$

Example 4.1 (subspace metric). If (X, d_X) is a metric space and $E \subseteq X$ is any subset, then

$$d_E : E \times E \to \mathbb{R}_{\geq 0} \quad \text{where} \quad d_E(p_1, p_2) = d_X(p_1, p_2)$$

is the metric on E induced by X (or d_X).

Example 4.2 (discrete metric). For any set X, define

$$d : X \times X \to \mathbb{R} \quad \text{where} \quad d(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \neq x_2; \\ 0 & \text{if } x_1 = x_2. \end{cases}$$

Then, d is a metric on X, called the discrete metric.

Lemma 4.1 (uniqueness of limit in metric space). Let X be a metric space. If the limit of a sequence $\{x_n\}_{n \in \mathbb{N}}$ exists, then it is unique. That is to say,

$$\text{if } x, x' \in X \quad \text{are such that} \quad \lim_{n \to \infty} x_n = x \text{ and } \lim_{n \to \infty} x_n = x',$$

then $x = x'$ in X.

Definition 4.2 (eventually constant). Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in a metric space (X, d). We say that the sequence is eventually constant if and only if

$$\text{there exists } N \in \mathbb{N} \text{ such that for all } n \geq N \quad \text{we have} \quad x_n = x_N.$$

Definition 4.3 (boundedness). Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in a metric space (X, d). We say that the sequence is bounded if and only if there exists $M > 0$ and a point $x \in X$ such that

$$d(x_n, x) \leq M \quad \text{for all } n \in \mathbb{N}.$$

Recall Example 4.2, where we introduced the discrete metric.

Proposition 4.2. Suppose d is the discrete metric on a set X. Then, a sequence

$$\{x_n\}_{n\in\mathbb{N}} \text{ converges to } x \in X \quad \text{if and only if} \quad \text{it is eventually constant of value } x.$$

Proof. The reverse direction holds trivially. For the forward direction, suppose $\{x_n\}_{n\in\mathbb{N}} \to x$ in X, which is equipped with the discrete metric. Take $\varepsilon = \frac{1}{2}$. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$d(x_n, x) < \frac{1}{2} \quad \text{which implies} \quad x_n = x,$$

where we used the fact that d is the discrete metric. □

Lemma 4.2 (subsequence). The sequence

$$\{x_n\}_{n\in\mathbb{N}} \text{ converges in } X \quad \text{if and only if} \quad \text{every subsequence of } \{x_n\}_{n\in\mathbb{N}} \text{ converges in } X.$$

When this is so, the limit of $\{x_n\}_{n\in\mathbb{N}}$ is equal to the limit of any of its subsequences.

Definition 4.4. Let X be a metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is Cauchy if and only if

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \quad \text{such that} \quad \text{for all } m,n \geq N \text{ we have } d(x_m, x_n) < \varepsilon.$$

Proposition 4.3. Let X be a metric space. If

$$\{x_n\}_{n\in\mathbb{N}} \text{ is convergent in } X \quad \text{then} \quad \{x_n\}_{n\in\mathbb{N}} \text{ is Cauchy.}$$

Proposition 4.4. Let X be a metric space. If

$$\{x_n\}_{n\in\mathbb{N}} \text{ is a Cauchy sequence} \quad \text{then} \quad \text{it is bounded.}$$

Proposition 4.5. Let X be a metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in X. If

$$\text{there exists a subsequence of } \{x_n\}_{n\in\mathbb{N}} \text{ that converges in } X \quad \text{then} \quad \{x_n\}_{n\in\mathbb{N}} \text{ also converges in } X,$$

and to the same limit.

Definition 4.5 (Cauchy completeness). A metric space X is

$$\text{Cauchy complete} \quad \text{if and only if} \quad \text{every Cauchy sequence in } X \text{ converges in } X.$$

Example 4.3 (Euclidean spaces). Let $V = \mathbb{R}^k$ be a Euclidean space with an ordered basis $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$. For all $1 \leq i \leq k$, let

$$\pi_i : V \to \mathbb{R} \quad \text{denote} \quad \text{the } i^{\text{th}} \text{ projection/coordinate map with respect to } \mathcal{B}.$$

A sequence

$$\{x_n\}_{n\in\mathbb{N}} \text{ in } V \text{ converges in } V \text{ to } x \in V \quad \text{if and only if} \quad \text{for all } 1 \leq i \leq k \text{ we have } \lim_{n\to\infty} \pi_i(x_n) = \pi_i(x).$$

To see why, first let

$$x_n = (\pi_1(x_n), \pi_2(x_n), \ldots, \pi_k(x_n)) \quad \text{and} \quad x = (\pi_1(x), \pi_2(x), \ldots, \pi_k(x)).$$

For the forward direction, suppose $x_n \to x$ in $\mathbb{R}^k$. By the definition of convergence in a normed space, this means that

$$\lim_{n\to\infty} \|x_n - x\| = 0,$$

where the norm $\|\cdot\|$ is given by the Euclidean norm

$$\|x_n - x\| = \sqrt{\sum_{i=1}^{k} (\pi_i(x_n) - \pi_i(x))^2}.$$

Now, for any fixed i with $1 \le i \le k$, we have

$$|\pi_i(x_n) - \pi_i(x)| \le \sqrt{\sum_{j=1}^{k} (\pi_j(x_n) - \pi_j(x))^2} = \|x_n - x\|.$$

Since $\|x_n - x\| \to 0$ as $n \to \infty$, then the forward direction holds.

For the reverse direction, assume that for each $1 \le i \le k$, we have

$$\lim_{n\to\infty} \pi_i(x_n) = \pi_i(x).$$

We want to show that $\lim_{n\to\infty} \|x_n - x\| = 0$. Given $\varepsilon > 0$, since each coordinate converges, there exists $N_i \in \mathbb{N}$ such that for all $n \ge N_i$,

$$|\pi_i(x_n) - \pi_i(x)| < \frac{\varepsilon}{\sqrt{k}}.$$

Let $N = \max\{N_1, N_2, \ldots, N_k\}$. Then for all $n \ge N$ and for every $1 \le i \le k$, we have

$$|\pi_i(x_n) - \pi_i(x)| < \frac{\varepsilon}{\sqrt{k}}.$$

Now, consider the Euclidean norm. Using the bound for each coordinate, we have:

$$\|x_n - x\| \le \sqrt{\sum_{i=1}^{k} \left(\frac{\varepsilon}{\sqrt{k}}\right)^2} = \sqrt{k \cdot \frac{\varepsilon^2}{k}} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\lim_{n\to\infty} \|x_n - x\| = 0,$$

which is precisely the definition of convergence in $\mathbb{R}^k$.

Corollary 4.1 (limit properties of convergent sequences in Euclidean spaces)**.** Let V be a Euclidean space, $\{k_n\}_{n\in\mathbb{N}}$ be a convergent sequence in $\mathbb{R}$, and $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ be convergent sequences in V. Then, the following hold:

(i) $\{x_n + y_n\}_{n\in\mathbb{N}}$ is also convergent in V and

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n \quad \text{in } V$$

(ii) $\{-x_n\}_{n\in\mathbb{N}}$ is also convergent in V and

$$\lim_{n\to\infty} -x_n = -\lim_{n\to\infty} x_n \quad \text{in } V$$

(iii) $\{k_n x_n\}_{n\in\mathbb{N}}$ is also convergent in V and

$$\lim_{n\to\infty} k_n x_n = \left(\lim_{n\to\infty} k_n\right)\left(\lim_{n\to\infty} x_n\right) \quad \text{in } V$$

Corollary 4.2. Any Euclidean space is Cauchy complete.

4.2 Maps between Metric Spaces

Definition 4.6 (isometry). Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is said to be an isometry if and only if

$$\text{for all } x_1, x_2 \in X \quad \text{one has} \quad d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) \text{ in } \mathbb{R}_{\geq 0}.$$

Proposition 4.6 (isometry implies injective). An isometry $f : X \to Y$ is injective.

Proof. Suppose $x_1, x_2 \in X$ are such that $f(x_1) = f(x_2)$ in Y. Then, by positive definiteness of d_Y, we have

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)) = 0 \quad \text{in } \mathbb{R}_{\geq 0}.$$

We conclude that $x_1 = x_2$ in X. □

From Proposition 4.6, we infer that most maps from a metric space (X, d_X) to $\mathbb{R}$ with the metric induced by $|\cdot|_{\mathbb{R}}$ are not isometries.

Definition 4.7 (continuity). Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a map. We say that f is continuous at $a \in X$ if and only if

$$\text{for every } \varepsilon > 0 \text{ there exists } \delta > 0 \quad \text{such that} \quad \text{for all } a \in X \text{ with } d_X(x, a) < \delta \text{ we have } d_Y(f(x), f(a)) < \varepsilon.$$

Also, f is continuous everywhere on X if and only if for all $a \in X$, f is continuous at a.

Definition 4.8 (continuous function). Let X be a metric space. A continuous function on X is

$$\text{a continuous map } f : X \to \mathbb{R} \quad \text{where} \quad \mathbb{R} \text{ is given by the metric induced on } |\cdot|_{\mathbb{R}}.$$

Example 4.4 (isometry implies continuity). If

$$f : X \to Y \text{ is an isometry} \quad \text{then} \quad f \text{ is continuous}.$$

This can be easily seen by setting $= \varepsilon$ in Definition 4.7.

Example 4.5. Any map $f : X \to \{a\}$ from a metric space X to a singleton is continuous.

Example 4.6. Any map $f : \{a\} \to X$ from a singleton to a metric space is continuous.

Example 4.7 (constant maps are continuous). Any constant map $f : X \to Y$ (i.e. there exists $y \in Y$ such that for all $x \in X$ we have $f(x) = y$) is continuous.

Example 4.8. Fix $q \in X$. Then,

$$\text{the real-valued function } d_q : X \to \mathbb{R} \quad \text{where} \quad d_q(p) = d(p, q) \quad \text{is continuous}.$$

To see why, let $p \in X$. Given $\varepsilon > 0$, take $\delta = \varepsilon > 0$. As such, for all x with $d(x,p) < \delta$, we have

$$d(x,q) \le d(x,p) + d(p,q) < \delta + d(p,q) \quad \text{and}$$
$$d(p,q) \le d(p,x) + d(x,q) < \delta + d(x,q)$$

where we used the triangle inequality. Hence,

$$\left|d_q(x) - d_q(p)\right| = |d(x,q) - d(p,q)| < \delta = \varepsilon$$

and the result follows.

Proposition 4.7 (sequential criterion for continuity)**.** Let (X,d_X) and (Y,d_Y) be metric spaces. Let $f : X \to Y$ be a map. Then,

$$f \text{ is continuous at } x \in X \quad \text{if and only if} \quad \begin{array}{l} \text{for every sequence } \{x_n\}_{n\in\mathbb{N}} \text{ with } x_n \to x \text{ in } X \\ \text{one has } \{f(x_n)\}_{n\in\mathbb{N}} \to f(x) \text{ in } Y \end{array}$$

In other words, f preserves limits of convergent sequences.

Proof. We first prove the forward direction. Say f is continuous at $x \in X$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $\{x_n\}_{n\in\mathbb{N}} \to x$ in X. As such, we obtain a sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ in Y. Given $\varepsilon > 0$, by continuity of f, there exists $\delta > 0$ such that

$$\text{for any } x_0 \in X \text{ with } d_X(x_0,x) < \delta \quad \text{one has} \quad d_Y(f(x_0), f(x)) < \varepsilon.$$

Since $\{x_n\}_{n\in\mathbb{N}} \to x$ in X, then there exists $N \in \mathbb{N}$ such that for all $n \ge N$, we have

$$d_X(x_n,x) < \delta \quad \text{so} \quad d_Y(f(x_n), f(x)) < \varepsilon.$$

For the forward direction, suppose on the contrary that f is not continuous at $x \in X$. Then, there exists $\varepsilon > 0$ such that

$$\text{for any } \delta > 0 \text{ there exists } x_0 \in X \text{ with } d_X(x_0,x) < \delta \quad \text{and} \quad d_Y(f(x_0), f(x)) \ge \varepsilon.$$

For each $n \in \mathbb{N}$, we apply the above condition with $\delta = \frac{1}{n} > 0$. We also choose

$$x_n \in X \quad \text{with} \quad d_X(x_n,x) < \frac{1}{n} \text{ and } d_Y(f(x_n), f(x)) \ge \varepsilon.$$

Then, $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X, and since $(d_X(x_n,x))_{n\in\mathbb{N}} \to 0$, we see that $x_n \to x$ in X. However, since for all $n \in \mathbb{N}$, we have $d_Y(f(x_n), f(x)) \ge \varepsilon$, then the sequence $f(x_n)$ does not tend to $f(x)$ in Y. □

Example 4.9 (Bartle and Sherbert p. 134 Question 12)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be additive if

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Prove that if f is continuous at some point x_0, then it is continuous at every point of $\mathbb{R}$.

Solution. By the given functional equation, we can let $x = y = 0$ so $f(0) = 2f(0)$. As such, $f(0) = 0$. We shall prove that f is continuous at 0, so consider

$$f(h) = f(x_0 + h) - f(x_0).$$

Letting $h \to 0$, we have

$$\lim_{h\to 0} f(h) = \lim_{h\to 0} [f(x_0+h) - f(x_0)] = 0.$$

So, f is continuous at 0. Lastly, we prove that f is continuous at any point of $\mathbb{R}$. Let $a \in \mathbb{R}$ be arbitrary. Then,

$$f(h) = f(a+h) - f(a).$$

Since

$$\lim_{h\to 0} f(h) = 0 \quad \text{then} \quad \lim_{h\to 0} f(a+h) = f(a)$$

and the result follows. □

Example 4.10 (Bartle and Sherbert p. 134 Question 13). Suppose that f is a continuous additive function on $\mathbb{R}$. If $c = f(1)$, show that we have $f(x) = cx$ for all $x \in \mathbb{R}$.
Hint: First show that if r is a rational number, then $f(r) = cr$.

Solution. Let $r \in \mathbb{Q}$. Then, there exist $m, n \in \mathbb{Z}$, with $n \neq 0$, such that $r = \frac{m}{n}$. So,

$$f(nr) = f\left(n \cdot \frac{m}{n}\right) = f(m) = mf(1) = mc.$$

By applying the additive property on nr, we have

$$f(nr) = nf(r) \quad \text{so} \quad nf(r) = mc.$$

As such, $f(r) = cr$. We then prove that $f(x) = cx$ for all $x \in \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists a sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ which converges to x. By continuity of f, we have

$$f(x) = f\left(\lim_{n\to\infty} r_n\right) = \lim_{n\to\infty} f(r_n).$$

Since $f(r) = cr$ for any rational number $r \in \{r_n\}_{n\in\mathbb{N}}$, it follows that $f(x) = cx$. □

Proposition 4.8 (identity map is continuous). The identity map

$$\mathrm{id}_X : X \to X \quad \text{where} \quad x \mapsto x \quad \text{is continuous.}$$

Proof. This is obvious — take $\delta = \varepsilon$ in the definition of continuity (Definition 4.7). □

Proposition 4.9 (composition of continuous functions is continuous). Let

$$f : X \to Y \quad \text{and} \quad g : Y \to Z \quad \text{be maps between metric spaces.}$$

If f is continuous at $x \in X$ and g is continuous at $f(x) \in Y$, then $g \circ f$ is continuous at $x \in X$. Thus, if

$$f \text{ and } g \text{ are continuous} \quad \text{then} \quad g \circ f \text{ is also continuous.}$$

Proof. Use sequential criterion for continuity (Proposition 4.7). □

Proposition 4.10 (universal property of the product topology). Let X and Y be metric spaces. Let $f, g : X \to Y$ be maps. Then, the map

$$(f, g) : X \to Y \times Y \quad \text{where} \quad x \mapsto (f(x), g(x)) \quad \text{is continuous}$$

if and only if both f and g are continuous (Figure 11).

Proof. For the forward direction, from Figure 11, note that the maps

$$\pi_1 : Y \times Y \to Y \quad \text{and} \quad (\pi_2 : Y \times Y \to Y$$

are the canonical projection maps, and by definition of the metric product, they are continuous.

Since π_1 is continuous and the diagram commutes (i.e., $\pi_1 \circ (f,g) = f$), the composition $f = \pi_1 \circ (f,g)$ is continuous. Similarly, since π_2 is continuous and $\pi_2 \circ (f,g) = g$, the composition $g = \pi_2 \circ (f,g)$ is continuous.

For the reverse direction, the universal property of the product (illustrated by Figure 11) states that a map $h : X \to Y \times Y$ is continuous if and only if the compositions with the projection maps, $\pi_1 \circ h$ and $\pi_2 \circ h$, are continuous. Here, set $h = (f,g)$. We already know that f and g are continuous by assumption. Hence, $\pi_1 \circ (f,g) = f$ and $\pi_2 \circ (f,g) = g$ are continuous, and the result follows. $\square$

$$\begin{tikzcd}
 & & Y \\
X \arrow[urr, "f"] \arrow[r, "(f,g)"] \arrow[drr, "g"'] & Y \times Y \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & & Y
\end{tikzcd}$$

Figure 11: Universal property of the product topology

Example 4.11. Consider the metric spaces $X = \mathbb{R}$ and $Y = \mathbb{R}$. Define

$$f(x) = x \quad \text{and} \quad g(x) = x^2.$$

Both f and g are continuous functions. According to Proposition 4.10, the map

$$(f,g) : \mathbb{R} \to \mathbb{R} \times \mathbb{R} \quad \text{where} \quad x \mapsto (x, x^2) \quad \text{is continuous.}$$

Proposition 4.11. Let X and Y be metric spaces. so that operations such as addition, subtraction, multiplication and scalar multiplication are defined and continuous. Suppose that

$$f, g : X \to Y \text{ are functions} \quad \text{that} \quad \text{are continuous at a point } a \in X.$$

Then, we can rigorously define the following functions:

(i) Sum and difference: Define

$$\begin{aligned} f+g : X \to Y \quad &\text{where} \quad x \mapsto f(x) + g(x) \quad \text{and} \\ f-g : X \to Y \quad &\text{where} \quad x \mapsto f(x) - g(X) \end{aligned}$$

Since the addition (and subtraction) map $+ : Y \times Y \to Y$ is continuous, the composition

$$x \mapsto (f(x), g(x)) \mapsto f(x) + g(x) \quad \text{is continuous at } a.$$

(ii) Scalar multiplication: For any fixed scalar $\alpha \in \mathbb{R}$, define

$$\alpha f : X \to Y \quad \text{where} \quad x \mapsto \alpha f(x).$$

The continuity of the scalar multiplication operation on Y guarantees that αf is continuous at a.

(iii) Product: Define

$$fg : X \to Y \quad \text{where} \quad x \mapsto f(x)g(x).$$

The continuity of the multiplication map $Y \times Y \to Y$ implies that fg is continuous at a.

(iv) Quotient: If $g(a) \neq 0$, then there exists a neighbourhood of a where $g(x) \neq 0$. Define

$$\frac{f}{g} : \{x \in X : g(x) \neq 0\} \to Y \quad \text{where} \quad x \mapsto \frac{f(x)}{g(x)}.$$

Using the continuity of g, $\frac{f}{g}$ is continuous at a.

Example 4.12 (Bartle and Sherbert p. 129 Question 3)**.** Let $a < b < c$. Suppose that

$$f \text{ is continuous on } [a,b] \quad \text{and that} \quad g \text{ is continuous on } [b,c],$$

and that $f(b) = g(b)$. Define h on $[a,c]$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in [a,b]; \\ g(x) & \text{if } x \in [b,c]. \end{cases}$$

Prove that h is continuous on $[a,c]$.

Solution. This is also known as the *pasting lemma*. We first note that because f is continuous on $[a,b]$, then it is continuous at every point $x \in [a,b]$. The same claim can be made for g. As such, $h(x) = f(x)$ is continuous at every point $x \in [a,b)$, and $h(x) = g(x)$ is continuous at every point $x \in (b,c]$.

It now suffices to show that h is continuous at $x = b$. Let $\varepsilon > 0$ be arbitrary. Since f is continuous at $x = b$, then whenever $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|x - b| < \delta \quad \text{implies} \quad |f(x) - f(b)| < \varepsilon.$$

Similarly, since g is continuous at $x = b$, then whenever $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$|x - b| < \delta_2 \quad \text{implies} \quad |g(x) - g(b)| < \varepsilon.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $|x - b| < \delta$, we have

$$|h(x) - h(b)| = |f(x) - f(b)| < \varepsilon \quad \text{if } x \in [a,b].$$

A similar argument holds for the case when $x \in [b,c]$, so the result follows. □

Example 4.13 (Bartle and Sherbert p. 134 Question 15)**.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous at a point c, and let

$$h(x) = \sup\{f(x), g(x)\} \quad \text{for } x \in \mathbb{R}.$$

Show that

$$h(x) = \frac{f(x) + g(x)}{2} + \left| \frac{f(x) - g(x)}{2} \right|$$

for all $x \in \mathbb{R}$. Use this to show that h is continuous at c.

Solution. For the first part, first consider the case where $f(x) \geq g(x)$. Then, $\sup\{f(x), g(x)\} = f(x)$. This implies that

$$\frac{f(x)+g(x)}{2} + \left|\frac{f(x)-g(x)}{2}\right| = \frac{f(x)+g(x)}{2} + \frac{f(x)-g(x)}{2} = f(x) \quad \text{since } f(x) \geq g(x).$$

This implies that $h(x) = f(x)$.

We then consider the case where $f(x) < g(x)$. Then, $\sup\{f(x), g(x)\} = g(x)$. This implies that

$$\frac{f(x)+g(x)}{2} + \left|\frac{f(x)-g(x)}{2}\right| = \frac{f(x)+g(x)}{2} - \frac{f(x)-g(x)}{2} = g(x) \quad \text{since } f(x) < g(x).$$

This implies that $h(x) = g(x)$.

For the second part, we note that because $f, g : \mathbb{R} \to \mathbb{R}$ are continuous at c, then their sum and difference also continuous at c. That is to say,

$$f+g, f-g : \mathbb{R} \to \mathbb{R} \quad \text{are} \quad \text{continuous at } c.$$

So,

$$\frac{f(x)+g(x)}{2} \text{ and } \frac{f(x)-g(x)}{2} \quad \text{are} \quad \text{continuous at } c.$$

If a function f is continuous at c, then $|f|$ is also continuous at c, so it follows that h is continuous at c. □

Definition 4.9 (ε-neighbourhood of a point)**.** Let $a \in \mathbb{R}$ and $\varepsilon > 0$. The ε-neighbourhood of a is

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x-a| < \varepsilon\}.$$

Example 4.14 (Bartle and Sherbert p. 129 Question 6)**.** Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be continuous at a point $c \in A$. Show that for any $\varepsilon > 0$, there exists a neighbourhood $V_\delta(c)$ of c such that

$$\text{if } x, y \in A \cap V_\delta(c) \quad \text{then} \quad |f(x) - f(y)| < \varepsilon$$

Solution. Since f is continuous at c, then for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x-c| < \delta$, we have

$$|f(x) - f(c)| < \frac{\varepsilon}{2}.$$

Let $x, y \in A \cap V_\delta(c)$, where $V_\delta(c)$ consists of all $x \in \mathbb{R}$ such that $|x-c| < \delta$. By the triangle inequality, we have

$$|f(x) - f(y)| < |f(x) - f(c)| + |f(y) - f(c)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

which is equal to ε. The result follows. □

Example 4.15 (Bartle and Sherbert p. 129 Question 7)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at c and suppose $f(c) > 0$. Show that there exists a neighbourhood $V_\delta(c)$ of c such that $f(x) > 0$ for all $x \in V_\delta(c)$.

Solution. Since f is continuous at c, then for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{whenever } |x-c| < \delta \quad \text{we have} \quad |f(x) - f(c)| < \varepsilon.$$

In particular, choose $\varepsilon = \frac{f(c)}{2}$ so

$$\frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}.$$

In particular, for every x in $V_\delta(c)$, we must have $f(x) > 0$. □

Definition 4.10 (floor function)**.** The floor function of a number x, which is denoted by $\lfloor x \rfloor$, is defined to be the greatest integer less than or equal to x. Hence, for $n \in \mathbb{Z}$,

$$\lfloor x \rfloor = n \quad \text{if } x \in [n, n+1).$$

Example 4.16. $\lfloor \pi \rfloor = 3$ and $\lfloor -4.8 \rfloor = -5$

Example 4.17 (Bartle and Sherbert p. 129 Question 4)**.** If $x \in \mathbb{R}$, define $[[x]]$ to be the greatest integer $n \in \mathbb{Z}$ such that $n < x$. (For example, $[[8.3]] = 8$ and $[[-\pi]] = -4$.) The function $x \mapsto [[x]]$ is called the greatest integer function (also known as floor function). Determine the points of continuity of each of the following functions:

(a) $f(x) = [[x]]$, **(c)** $h(x) = [[\sin x]]$,
(b) $g(x) = x[[x]]$ **(d)** $k(x) = \left[\left[\frac{1}{x}\right]\right]$, where $x \neq 0$

Solution.
(a) Continuous for all $x \in (n, n+1)$, where $n \in \mathbb{Z}$.
(b) Continuous for all $x \in (n, n+1) \cup \{0\}$, where $n \in \mathbb{Z}$. This is because

$$g(0) = 0 \text{ by direct substitution} \quad \text{and} \quad \lim_{x \to 0^-} g(x) = 0.$$

(c) Continuous for all $x \in (n\pi, (n+1)\pi)$, where $n \in \mathbb{Z}$.
(d) Continuous for all

$$x \in \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n} \right) \cup \bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, -\frac{1}{n+1} \right) \cup (-\infty, -1) \cup (1, \infty).$$

□

Definition 4.11 (fractional part)**.** For any number x, the fractional part of it is defined by $\{x\}$. So, for any $x > 0$, we have

$$\{x\} = x - \lfloor x \rfloor.$$

Example 4.18 (Bartle and Sherbert p. 134 Question 4)**.** Let $x \mapsto [[x]]$ denote the greatest integer function. Determine the points of continuity of the function $f(x) = x - [[x]]$.

Solution. We claim that f is continuous on all $x \in \mathbb{R} \setminus \mathbb{Z}$. The function f in the question is also known as the fractional part function, which returns the fractional part of x.

For any $n \in \mathbb{Z}$, on the interval $(n, n+1)$, we note that $[[x]]$ is constant and equal to n. Hence, on $(n, n+1)$ the function simplifies to $f(x) = x - n$, which is a linear function. Linear functions are continuous, so f is continuous on each interval $(n, n+1)$.

At the integer points, the behaviour of $f(x)$ changes because the value of the greatest integer function (or floor function) jumps. For any integer n, we have

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} (x - n) = 0 \quad \text{but} \quad \lim_{x \to n^-} f(x) = \lim_{x \to n^-} (x - (n-1)) = 1.$$

As the left-hand and right-hand limits at $x = n$ are different, the function is discontinuous at every $n \in \mathbb{Z}$. □

Definition 4.12 (ceiling function). The ceiling function of a number x, which is denoted by $\lceil x \rceil$, is defined to be the least integer greater than or equal to x. Hence, for $n \in \mathbb{Z}$,

$$\lceil x \rceil = n \quad \text{if } x \in (n, n+1].$$

Example 4.19. $\lceil 6.1 \rceil = 7$ and $\lceil -7.8 \rceil = -7$

Two important inequalities in relation to the floor and ceiling function respectively are

$$n \leq \lfloor x \rfloor < n+1 \text{ and } n < \lceil x \rceil \leq n+1 \quad \text{for } n \in \mathbb{Z},$$

which can be used to solve equations, inequalities and limits involving them.

Example 4.20 (Bartle and Sherbert p. 129 Question 11). Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition

$$|f(x) - f(y)| < K|x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

Show that f is continuous at every point $c \in \mathbb{R}$.

Solution. The given condition is known as *Lipschitz continuity*. Let $\varepsilon > 0$ be arbitrary. Choose $\delta = \frac{\varepsilon}{K}$ Then, whenever $|x - c| < \delta$, we have

$$|f(x) - f(c)| < \frac{\varepsilon}{\delta}|x - c| < \frac{\varepsilon}{\delta} \cdot \delta = \varepsilon$$

so f is continuous at every point $c \in \mathbb{R}$. □

Example 4.21 (Bartle and Sherbert p. 129 Question 12). Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at every point of $\mathbb{R}$ and that $f(r) = 0$ for every rational r. Prove that $f(x) = 0$ for every $x \in \mathbb{R}$.

Solution. Suppose on the contrary that there exists $x \in \mathbb{R}$ such that $f(x) \neq 0$. Without loss of generality, say $f(x) > 0$. Since f is continuous at x, then there exists $\delta > 0$ such that for any y satisfying $|y - x| < \delta$, we have

$$|f(y) - f(x)| < \frac{f(x)}{2}.$$

This implies that for any $y \in (x - \delta, x + \delta)$, we have $f(y) > 0$. However, by the density theorem (Theorem 1.1), $\mathbb{Q}$ is dense in $\mathbb{R}$ so there exists at least one rational number r in $(x - \delta, x + \delta)$. By the hypothesis, $f(r) = 0$, which is a contradiction as we must have $f(r) > 0$. □

Example 4.22 (Bartle and Sherbert p. 130 Question 14). Let $A = (0, 1)$, and let $k : A \to \mathbb{R}$ be defined as follows. For $x \in A$, if x is irrational, we define $k(x) = 0$; for $x \in A$ rational and of the form $x = \frac{m}{n}$ with natural numbers m, n having no common factors except 1, we define $k(x) = n$.

Prove that k is unbounded on every open interval in A. Conclude that k is not continuous at any point of A.

Solution. Let I be an arbitrary open interval in $(0, 1)$. We will prove that for any positive integer N, there exists a rational number $x \in I$ of the form $x = \frac{m}{n}$ in lowest terms with $n > N$. One way to go about is to consider Farey sequences. For a given N, the Farey sequence of order N (Figure 12) contains all rationals in $[0, 1]$ with denominators at most N. The gaps between these rationals become arbitrarily small as N increases. Since I is an open interval, it will eventually contain rationals that are not in the Farey sequence of any fixed order N. In other words, there is a rational $\frac{m}{n} \in I$ in lowest terms for which $n > N$.
Since $N \in \mathbb{N}$ was arbitrary, this shows that k is unbounded on I.

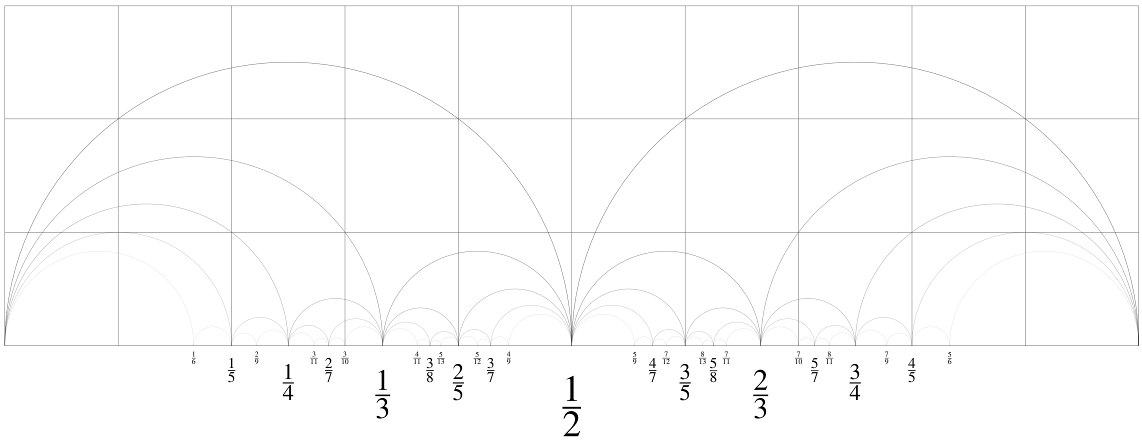


Figure 12: Visualising a Farey sequence

For the second point, suppose $c \in A$ is irrational. At c, we have $k(c) = 0$. However, in any open interval containing c, there exist rational numbers with arbitrarily large denominators, hence with arbitrarily large values of q. As such, we can take a sequence x_n of rational numbers converging to c. Then, the sequence $k(x_n)$ can be made to diverge to infinity, contradicting the requirement for continuity.

On the other hand, suppose c is rational. At c, suppose $c = \frac{p}{q}$ in lowest terms, so $k(c) = q$. Again, any neighbourhood of c contains irrational numbers x, for which $k(x) = 0$. Consider a sequence of irrational numbers converging to c. Then, $k(x_n) = 0$ for all n, so the limit of $k(x_n) = 0$, which is different from $k(c) = q$. The result follows. $\square$

Example 4.23 (Bartle and Sherbert p. 134 Question 7)**.** Give an example of a function $f : [0,1] \to \mathbb{R}$ that is discontinuous at every point of $[0,1]$ but such that $|f|$ is continuous on $[0,1]$.

Solution. Consider

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}' \\ -1 & \text{if } x \in \mathbb{Q} \end{cases}$$

so that $|f(x)| = 1$, which holds for all $x \in \mathbb{R}$ since $\mathbb{R}$ is the disjoint union of rationals and irrationals. $\square$

Example 4.24 (Bartle and Sherbert p. 134 Question 8)**.** Let f, g be continuous from $\mathbb{R}$ to $\mathbb{R}$, and suppose $f(r) = g(r)$ for all rational numbers r. Is it true that $f(x) = g(x)$ for all $x \in \mathbb{R}$?

Solution. We claim that the statement is true. Suppose on the contrary that there exists $x \in \mathbb{R}$ such that $f(x) \neq g(x)$. Define

$$\varepsilon = \left| \frac{f(x) - g(x)}{2} \right| > 0.$$

As $f : \mathbb{R} \to \mathbb{R}$ is continuous at r, there exists $\delta_1 > 0$ such that whenever $|x - r| < \delta_1$, we have $|f(x) - f(r)| < \varepsilon$. Similarly, as $g : \mathbb{R} \to \mathbb{R}$ is continuous at r, there exists $\delta_2 > 0$ such that whenever $|x - r| < \delta_2$, we have $|g(x) - g(r)| < \varepsilon$. Take $\delta = \min\{\delta_1, \delta_2\}$. Then,

$$\begin{aligned} 2\varepsilon &= |f(x) - g(x)| \\ &= |f(x) - f(r) + f(r) + g(r) - g(x) - g(r)| \\ &\leq |f(x) - f(r)| + |g(x) - g(r)| + |f(r) - g(r)| \quad \text{by the triangle inequality} \\ &< \varepsilon + \varepsilon + |f(y) - g(y)| \end{aligned}$$

Choose $y \in (x-\delta, x+\delta)$ to be a rational number. Then, because $f(y) = g(y)$ for all $y \in \mathbb{Q}$, then $f(y) - g(y) = 0$, so it follows that $2\varepsilon < 2\varepsilon$, which is a contradiction. We conclude that $f(x) = g(x)$ for all $x \in \mathbb{R}$. $\square$

Example 4.25 (Bartle and Sherbert p. 134 Question 9). Let $h : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ satisfying $h\left(\frac{m}{2^n}\right) = 0$ for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Show that

$$h(x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

Solution. Suppose on the contrary that there exists $x \in \mathbb{R}$ such that $h(x) \neq 0$. Without loss of generality, say $h(x) > 0$. Since $h : \mathbb{R} \to \mathbb{R}$ is continuous at x, then there exists $\delta > 0$ such that for every $y \in \mathbb{R}$ satisfying

$$|y-x| < \delta \quad \text{we have} \quad |h(y) - h(x)| < \frac{h(x)}{2}.$$

In particular, we have

$$h(y) > \frac{h(x)}{2} > 0.$$

Note that the set of $\frac{m}{2^n}$ over all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ is dense in $\mathbb{R}$. In particular, we have $h(y_0) = 0$, where $y_0 = \frac{m}{2^n}$ is some rational[†]. Since y_0 was chosen such that $y_0 \in (x-\delta, x+\delta)$, then our continuity argument implies

$$h(y_0) > \frac{h(x)}{2} > 0$$

which is a contradiction since we must have $h(y_0) = 0$. We conclude that $h(x) = 0$ for all $x \in \mathbb{R}$. $\square$

4.3
Basic Results on Continuous Functions

Theorem 4.1 (extreme value theorem). Let $X = [a,b] \subseteq \mathbb{R}$ be a closed and bounded interval in $\mathbb{R}$. Let $f : X \to \mathbb{R}$ be a continuous function on X. Then,

$$\text{there exists } x \in X \quad \text{such that} \quad f(x) = \sup(f(X)) \text{ in } \mathbb{R}.$$

In the extreme value theorem (Theorem 4.1), $\sup f(X) \in \mathbb{R}$ is a finite real number, i.e. not $+\infty$. That is to say, f is bounded on X and is a value actually attained by f at the point $x \in X$. We make some remarks:

(i) The continuity of f is necessary. For example,

$$f : [-1,1] \to \mathbb{R} \quad \text{where} \quad f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0 \end{cases}$$

is not continuous, so $\sup(f[-1,1]) = +\infty$ is not attained by f.

(ii) Also, boundedness of X is necessary. For example,

$$f : [0,\infty) \to \mathbb{R} \quad \text{where} \quad f(x) = x$$

is unbounded so $\sup(f[0,\infty)) = +\infty$ is not attained by f.

(iii) The closedness of X is necessary. For example, we have

$$f : (0,1] \to \mathbb{R} \quad \text{where} \quad f(x) = \frac{1}{x}$$

is not closed so $\sup((0,1]) = +\infty$ is not attained by f.

[†] To be precise, y_0 is said to be a dyadic rational. Look up dyadic partitioning.

(iv) The least upper bound property of $\mathbb{R}$ is necessary. For example, we have

$$f:[0,2]\cap\mathbb{Q}\to\mathbb{R} \quad \text{where} \quad f(x)=\frac{1}{x^2-2}$$

and we note the domain does not satisfy the least upper bound property of $\mathbb{R}$ so $\sup(f[0,2]\cap\mathbb{Q})=+\infty$ is not attained by f.

We now prove the extreme value theorem (Theorem 4.1).

Proof. Suppose $f:X=[a,b]\to\mathbb{R}$ is continuous. We wish to show that there exists $x\in X$ such that $f(x)=\sup f(X)$ in $\mathbb{R}$. We already have $f(X)\subseteq\mathbb{R}$ since $X\neq\emptyset$, so $\sup f(X)$ exists in $\mathbb{R}\cup\{+\infty\}$.

We claim that $\sup f(X)$ lies in $\mathbb{R}$, i.e. not $+\infty$. Suppose on the contrary that $\sup f(X)=+\infty$. Then,

$$\text{for all } n\in\mathbb{N} \text{ there exists } x_n\in X \quad \text{such that} \quad f(x_n)\geq n \text{ in } \mathbb{R}.$$

Thus, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X\subseteq\mathbb{R}$ such that $f(x_n)\to\infty$ in $[-\infty,\infty]$. By the monotone subsequence theorem (Theorem 2.15), there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ which is monotone. Since X is a bounded set and by the least upper bound property of $\mathbb{R}$, the monotone subsequence must converge in $\mathbb{R}$, i.e.

$$\text{there exists } x\in\mathbb{R} \quad \text{such that} \quad x_{n_k}\to x \text{ as } k\to\infty.$$

Since X is a closed interval, then $x\in X$. By continuity of f, we have $f(x_{n_k})\to f(x)$ in $\mathbb{R}$, which is a contradiction because we earlier assumed that $f(x_n)\to\infty$.

We then claim that

$$\text{there exists } x\in X \quad \text{such that} f(x)=\sup f(X) \text{ in } \mathbb{R}.$$

By definition of supremum,

$$\text{for each } n\in\mathbb{N} \quad \text{we have} \quad \sup f(X)-\frac{1}{n}\in\mathbb{R} \text{ is not an upper bound of } f(X).$$

So,

$$\text{there exists } x_n\in X \quad \text{such that} \quad \sup f(X)-\frac{1}{n}<f(x_n)\leq\sup f(X) \text{ in } \mathbb{R}.$$

Thus, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X\subseteq\mathbb{R}$. By the Archimedean property of $\mathbb{R}$, $f(x_n)\to\sup f(X)$ in $\mathbb{R}$. By the monotone subsequence theorem (Theorem 2.15), there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ which is monotone. Since X is a bounded interval and by the least upper bound property of $\mathbb{R}$, this monotone subsequence converges in $\mathbb{R}$, i.e.

$$\text{there exists } x\in\mathbb{R} \quad \text{such that} \quad \{x_{n_k}\}_{k\in\mathbb{N}}\to x \text{ as } k\to\infty.$$

Since X is a closed interval, then $x\in X$. By continuity of f, we have $f(x_{n_k})\to f(x)$ in $\mathbb{R}$. By uniqueness of limits (Theorem 2.1), we have $f(p)=\sup f(X)$ in $\mathbb{R}$ and the result follows. □

Example 4.26 (Bartle and Sherbert p. 140 Question 1)**.** Let $I=[a,b]$ and let $f:I\to\mathbb{R}$ be a continuous function such that $f(x)>0$ for each x in I. Prove that there exists

$$\text{a number } \alpha>0 \quad \text{such that} \quad f(x)\geq\alpha \text{ for all } x\in I.$$

Solution. Since f is continuous on the closed interval $[a,b]$, by the extreme value theorem, it attains both a maximum and a minimum on $[a,b]$. In other words, there exists $x_0\in[a,b]$ such that

$$f(x_0)=\min\{f(x):x\in[a,b]\}.$$

Since $f(x)>0$ for all $x\in[a,b]$, then $\alpha=f(x_0)>0$. Thus, for any $x\in[a,b]$, we have $f(x)\geq\alpha$. □

Example 4.27 (Bartle and Sherbert p. 140 Question 13)**.** Suppose that

$$f : \mathbb{R} \to \mathbb{R} \text{ is continuous on } \mathbb{R} \quad \text{and} \quad \lim_{x\to-\infty} f(x) = 0 \text{ and } \lim_{x\to\infty} f(x) = 0.$$

Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or minimum on $\mathbb{R}$. Give an example to show that both a maximum and a minimum need not be attained.

Solution. We first prove that f is bounded on $\mathbb{R}$. By the formal definition of a limit, for every $\varepsilon > 0$, there exists $M > 0$ such that whenever $x > M$, we have $|f(x)| < \varepsilon$. In particular, we can choose $\varepsilon = 1$. Next, since f is continuous on $\mathbb{R}$, it is also continuous on the compact interval $[-M, M]$ (consider both limits). By the extreme value theorem (Theorem 4.1), f is bounded on $[-M, M]$, so f is bounded on $\mathbb{R}$, and the result follows.

Indeed, both a maximum and a minimum need not be attained as seen from the function $f(x) = e^{-x^2}$ which only has a maximum point at $x = 0$. □

Theorem 4.2 (intermediate value theorem)**.** Let $X \subseteq \mathbb{R}$ be any interval in $\mathbb{R}$. Let $f : X \to \mathbb{R}$ be a continuous function on X. Suppose

$$a, b \in X \quad \text{such that} \quad f(a) \le f(b) \text{ in } \mathbb{R}.$$

Then, for all $t \in \mathbb{R}$ (*secretly connoting the intermediate value*) with

$$f(a) \le t \le f(b) \quad \text{there exists } p \in X \text{ such that } f(p) = t \text{ in } \mathbb{R} \text{ (Figure 13)}.$$

Again, we make some remarks regarding the intermediate value theorem (Theorem 4.2).

(i) The continuity of f is necessary. To see why, suppose

$$f : [-1, 1] \to \mathbb{R} \quad \text{where} \quad f(x) = \begin{cases} 2 & \text{if } x \ge 0; \\ -2 & \text{if } x < 0 \end{cases}$$

which is not a continuous function. Then, f does not attain the intermediate values between -2 and 2.

(ii) Next, X must be an interval. Suppose otherwise, then for example, we have

$$f : [-1, 1] \setminus \{0\} \to \mathbb{R} \quad \text{where} \quad f(x) = x$$

is not an interval so f does not attain the intermediate value 0.

(iii) Lastly, the least upper bound property of $\mathbb{R}$ is necessary. To see why, consider

$$f : [0, 2] \cap \mathbb{Q} \to \mathbb{R} \quad \text{where} \quad f(x) = x^2$$

which is a continuous function but it does not attain the intermediate value 2 between 0 and 4.

We now prove the intermediate value theorem (Theorem 4.2).

Proof. Suppose we have an interval $X \subseteq \mathbb{R}$ and a continuous function $f : X \to \mathbb{R}$. Fix $a, b \in X$ and $t \in \mathbb{R}$ such that $f(a) \le t \le f(b)$. We wish to show that there exists $p \in X$ such that $f(x) = t$ in $\mathbb{R}$. Without loss of generality, one may assume that $a \le b$ in $X \subseteq \mathbb{R}$.

Consider

$$E = \{x \in [a, b] : f(x) \le t\} \subseteq \mathbb{R}.$$

Since $f(a) \le t$, then $a \in E$, so $E \ne \emptyset$. Also, E is bounded above by b. By the least upper bound property of $\mathbb{R}$, there exists $p = \sup E$ in $\mathbb{R}$. Then, $a \le p = \sup(E) \le b$. Since X is an interval, then we have $[a, b] \subseteq X$, so $p \in X$.

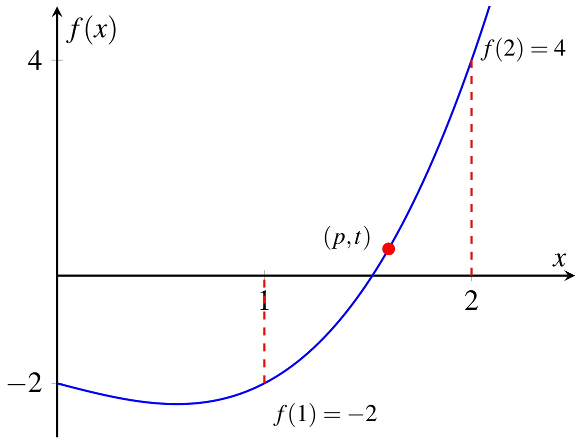


Figure 13: Graphical interpretation of the intermediate value theorem

We claim that $f(p) = t$. For each $n \in \mathbb{N}$, note that $p - \frac{1}{n} \in \mathbb{R}$ is not an upper bound of E, so

$$\text{there exists } p_n \in E \quad \text{such that} \quad p - \frac{1}{n} < p_n \leq p \text{ in } \mathbb{R}.$$

Thus, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $E \subseteq [a,b] \subseteq X$, which implies that for all $n \in \mathbb{N}$, $f(p_n) \leq t$. By the Archimedean property of $\mathbb{R}$, we have $x_n \to x$ in X. By continuity of f, we have $f(x_n) \to f(x)$ in $\mathbb{R}$. Hence, $f(p) \leq t$ in $\mathbb{R}$.

Suppose on the contrary that $f(p) < t$. Then, $\varepsilon = t - f(p) > 0$. By continuity of f,

$$\text{there exists } \delta > 0 \text{ such that for all } x \in [a,b] \text{ with } |x - p| < \delta \quad \text{then} \quad |f(x) - f(p)| < \varepsilon.$$

As such, $f(x) + f(p) + \varepsilon$. So, $a \leq p \leq b$.

Suppose on the contrary that $p < b$, then we can choose $x \in [a,b[$ such that $p < x < p + \delta$. Then, $f(x) < t$, so $x \in E$ by definition of E, but $x > p = \sup(E)$, which is a contradiction. Also, if $p = b$, then

$$f(b) = f(p) < t \leq f(b)$$

which again, is a contradiction. Hence, we must have $f(p) = t$ in $\mathbb{R}$. □

Example 4.28 (Bartle and Sherbert p. 140 Question 4). Show that every polynomial of odd degree with real coefficients has at least one real root.

Solution. Let

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \quad \text{where } n \text{ is odd.}$$

If $a_n > 0$, then

$$\lim_{x\to\infty} p(x) = \infty \quad \text{since} \quad \text{the leading term dominates the other expressions.}$$

Also,

$$\lim_{x\to-\infty} p(x) = -\infty \quad \text{since} \quad n \text{ is odd.}$$

On the other hand, if $a_n < 0$,

$$\lim_{x\to\infty} p(x) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} p(x) = \infty.$$

By the intermediate value theorem (Theorem 4.2), $p(x)$ has at least one root in $\mathbb{R}$. □

Example 4.29 (MA2108 AY19/20 Sem 1). Let f be continuous on $[0,1]$ and $f(0)=f(1)$. Prove that for any positive integer n, there exists a $\zeta\in[0,1]$ such that

$$f\left(\zeta+\frac{1}{n}\right)=f(\zeta).$$

Solution. Define

$$g(x)=f\left(x+\frac{1}{n}\right)-f(x).$$

By the intermediate value theorem (Theorem 4.2), g does not experience a change in its polarity for all $x\in[0,1]$. Suppose on the contrary that this claim is false. Then, by the method of differences,

$$\sum_{i=1}^{n} g\left(1-\frac{i}{n}\right)=f\left(\frac{1}{n}\right)-f(0) \quad \text{so} \quad g(0)=f\left(\frac{1}{n}\right)-f(0).$$

Without a loss of generality, assume that $g(x)>0$ for all $x\in[0,1]$. Then, setting $n=1$, it implies that $f(1)-f(0)>0$, which is a contradiction! □

Example 4.30 (Bartle and Sherbert p. 140 Question 3). Let $I=[a,b]$ and let $f:I\to\mathbb{R}$ be a continuous function on I such that for each x in I there exists y in I such that $|f(y)|\le\frac{1}{2}|f(x)|$. Prove

$$\text{there exists a point } c\in I \quad \text{such that} \quad f(c)=0.$$

Solution. Suppose on the contrary that no such c exists. That is to say, without loss of generality, $f(c)>0$ for all $c\in[a,b]$, otherwise it would contradict the continuity of f.

Choose an arbitrary point $x_0\in[a,b]$. By the hypothesis, there exists $x_1\in[a,b]$ such that

$$|f(x_1)|\le\frac{1}{2}|f(x_0)|.$$

Similarly, there exists $x_2\in[a,b]$ satisfying

$$|f(x_2)|\le\frac{1}{2}|f(x_1)|\le\frac{1}{2^2}|f(x_0)|.$$

Inductively, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq[a,b]$ such that

$$|f(x_n)|\le\frac{1}{2^n}|f(x_0)| \quad \text{for all } n\in\mathbb{Z}_{\ge 0}.$$

By the Bolzano-Weierstrass theorem[†], $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence. Let $\{x_{n_k}\}_{k\in\mathbb{N}}$ be such that $x_{n_k}\to c$ for some $c\in[a,b]$. Since f is continuous, then

$$f(c)=\lim_{k\to\infty}f(x_{n_k})\le\lim_{k\to\infty}\frac{1}{2^{n_k}}f(x_0)=0.$$

It follows that $f(c)=0$. The existence of a point $c\in I$ such that $f(c)=0$ contradicts our assumption that $f(x)>0$ for all $x\in I$. Therefore, our initial assumption is false. □

4.4 Special Functions

[†]This inherently uses the Heine-Borel theorem since $I=[a,b]$ is a closed and bounded interval.

Definition 4.13 (Dirichlet function). Named after mathematician Peter Gustav Lejeune Dirichlet, the Dirichlet function, $f(x)$, is defined to be the following:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

It is an example of a function that is nowhere continuous.

Theorem 4.3. The Dirichlet function is nowhere continuous.

Proof. Suppose $x \in \mathbb{Q}$, so $f(x) = 1$. We show that f is discontinuous at x. Let $\delta > 0$ be arbitrary and $y \in \mathbb{Q}$ such that $|x-y| < \delta$. Choose $\varepsilon = 1/2$. Without a loss of generality, assume $x < y$. Since there exists $z \in \mathbb{Q}'$ such that $x < z < y$ (due to the density of the irrationals in the reals), then

$$|f(x) - f(z)| = |1-0| = 1 > \frac{1}{2} = \varepsilon.$$

In a similar fashion, we now consider the case where $x > y$. There exists $z' \in \mathbb{Q}'$ such that $y < z' < x$, so

$$|f(x) - f(z')| = |1-0| = 1 > 1/2 = \varepsilon.$$

Therefore, if $x \in \mathbb{Q}$, f is discontinuous at x. For the case where $x \in \mathbb{Q}'$, the proof is very similar. $\square$

Lemma 4.3. The Dirichlet function can be constructed as the double limit of a sequence of continuous function. That is,

$$f(x) = \lim_{m\to\infty} \lim_{n\to\infty} \cos^{2n}(m!\pi x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

We then discuss Thomae's function (Definition 4.14), which is named after Carl Johannes Thomae, and the function is also known as the popcorn function due to its nature.

Definition 4.14 (Thomae's function). Thomae's function can be defined as follows:

$$f : \mathbb{R} \to [0,1] \quad \text{where} \quad f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q}; \\ 1/q & \text{if } x = \frac{p}{q},\ p,q \in \mathbb{N} \text{ and } \gcd(p,q) = 1 \end{cases}$$

It is a well-known fact that Thomae's function is not continuous at all rational points but continuous at all irrational points.

4.5 Uniform Continuity

Definition 4.15 (uniform continuity). Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. f is uniformly continuous on I if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{for any } x,y \in I \quad |x-y| < \delta \text{ implies } |f(x) - f(y)| < \varepsilon.$$

Corollary 4.3. If a function is uniformly continuous on I, then it is continuous on I.

Example 4.31. We claim that $f(x) = x^2$ is uniformly continuous on $[0,1]$. To see why, let $\varepsilon > 0$ be arbitrary. Choose $\delta = \varepsilon/2$. For $x, y \in [0,1]$, suppose $|x-y| < \delta$. Then,

$$|f(x) - f(y)| = |x^2 - y^2| = |x+y||x-y| < 2 \cdot \delta = \varepsilon$$

and we are done.

Theorem 4.4. A function f is uniformly continuous on I if and only if f' is bounded.

It is worth noting that $f(x) = x^2$ is uniformly continuous on $[a,b]$ in general, where $a, b \in \mathbb{R}$, but it is not uniformly continuous on $\mathbb{R}$!

Example 4.32 (MA2108 AY19/20 Sem 1). Prove that the function $f(x) = \sqrt{x^2 - x + 1}$ is uniformly continuous on $[1, \infty)$[†].

Solution. Since

$$f'(x) = \frac{1}{2}\left(x^2 - x + 1\right)^{-1/2} \cdot (2x - 1) = \frac{2x-1}{2\sqrt{x^2 - x + 1}},$$

and noting that $x^2 - x + 1 > 0$ for all $x \in [1, \infty)$, as well as $|f'(x)| < 1$, by Theorem 4.4, the result follows. □

Theorem 4.5 (sequential criterion for uniform continuity). $f : I \to \mathbb{R}$ is uniformly continuous on I if and only if for any two sequences $x_n, y_n \in I$ such that

$$\text{if } \lim_{n\to\infty} (x_n - y_n) = 0 \quad \text{then} \quad \lim_{n\to\infty} [f(x_n) - f(y_n)] = 0.$$

Definition 4.16 (Lipschitz continuity). Let I be an interval and $f : I \to \mathbb{R}$ satisfies the Lipschitz condition on I. Then, there is $K > 0$ such that

$$|f(x) - f(y)| \le K|x - y|, \quad \text{for all } x, y \in I.$$

Theorem 4.6. If a function is Lipschitz continuous on I, then it is uniformly continuous on I.

Example 4.33. We verify that $f(x) = x^2$, in the interval $[0,1]$, satisfies the Lipschitz condition.

Solution. Since $f(x) - f(y) = x^2 - y^2$, then

$$\left|\frac{f(x) - f(y)}{x - y}\right| = \left|\frac{x^2 - y^2}{x - y}\right| = |x + y| \le 2,$$

and since $2 > 0$, $f(x) = x^2$, in $[0,1]$, is said to satisfy the Lipschitz condition. In other words, f is Lipschitz continuous. □

Theorem 4.7. If $f : I \to \mathbb{R}$ is uniformly continuous on I and x_n is Cauchy, then $f(x_n)$ is Cauchy.

If the function $f : (a,b) \to \mathbb{R}$ is uniformly continuous on (a,b), then $f(a)$ and $f(b)$ can be defined so that the extended function is continuous on $[a,b]$.

Example 4.34. Let $f : [1, \infty) \to \mathbb{R}$ be a uniformly continuous function. Prove that

$$\text{there exists } M > 0 \text{ such that for all } x \ge 1 \quad \text{we have } |f(x)| \le Mx.$$

[†]The original question had an error. It used $f(x) = \sqrt{x(x-1)}$, which is undefined at $x = 1$. Confirmed the change with one of the students.

Solution. Since f is uniformly continuous, then for every ε, there exists $\delta > 0$ such that for every $x, y \geq 1$, whenever $|x-y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$. Consider $1 \leq x_1 < x_2 < \ldots$, so by uniform continuity, we have

$$|x_{k+1} - x_k| < \delta \quad \text{implies} \quad |f(x_{k+1}) - f(x_k)| < \varepsilon \quad \text{for all } k \in \mathbb{N}.$$

Let $x_1 = 1$ and $x_n = x$. Then,

$$|f(x) - f(1)| = \left| \sum_{k=1}^{n-1} f(x_{k+1}) - f(x_k) \right| \leq \sum_{k=1}^{n-1} |f(x_{k+1}) - f(x_k)| < (n-1)\varepsilon,$$

where we used the triangle inequality. So,

$$f(1) - (n-1)\varepsilon < f(x) < f(1) + (n-1)\varepsilon$$

Without loss of generality, suppose $|f(1) - (n-1)\varepsilon| < |f(1) + (n-1)\varepsilon|$ so

$$|f(x)| < |f(1) + (n-1)\varepsilon| < |f(1)| + |(n-1)\varepsilon|$$

We can choose $\varepsilon = 1$ so

$$|f(x)| < |f(1)| + n - 1.$$

Note that the number of increments $n-1$ is at most $\lceil (x-1)/\delta \rceil$, so

$$|f(x)| < |f(1)| + \left\lceil \frac{(x-1)}{\delta} \right\rceil \leq |f(1)| + \frac{x}{\delta}.$$

$\square$

Chapter 5
Topology

5.1
Introduction

Definition 5.1 (topology). A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(i) $\emptyset, X \in \mathcal{T}$

(ii) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$

(iii) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$

Definition 5.2 (open set). Let X be a set. Then,

$$\text{a subset } U \subseteq X \text{ is an open set of } X \quad \text{if} \quad U \text{ belongs to the collection } \mathcal{T}.$$

Example 5.1 (discrete topology). Let $X = \{a, b\}$ be a set. If a space has the *discrete topology*, then every subset of a set is considered an open set, i.e. $\mathcal{T} = \mathcal{P}(X)$, the power set of X.

Example 5.2 (trivial topology). Again, let $X = \{a, b\}$ be a set. If a space has the *trivial topology*, then the only open sets are and the whole space, i.e. $\mathcal{T} = \{\emptyset, X\}$.

Note that albeit uninteresting, if $X = \emptyset$ or $X = \{a\}$ a singleton, then X has a unique topology, which is both discrete and trivial.

Example 5.3 (finite complement topology). The finite complement topology, $\mathcal{T}_f$, is defined as follows:

$$\mathcal{T}_f = \{\text{all subsets } U \text{ of } X \quad \text{such that} \quad X \setminus U \text{ is finite or all of } X\}.$$

Example 5.4 (intersection of topologies). Let $\mathcal{F} = \{\mathcal{T}_i\}_{i \in I}$ be a non-empty family of topologies on X, where I is some indexing set. Then, their intersection, denoted by

$$\bigcap_{i \in I} \mathcal{T}_i \quad \text{is} \quad \text{also a topology on } X.$$

Let $\mathcal{S} \subseteq \mathcal{P}(X)$ be any collection of subsets of X. Then, the family $\mathcal{F}_{\mathcal{S}}$ consisting of all topologies $\mathcal{T}$ on X with $\mathcal{S} \subseteq \mathcal{T}$ is always non-empty, i.e.

$$\mathcal{T}_{\mathcal{S}} = \{\text{all } \mathcal{T} \subseteq \mathcal{P}(X) : \mathcal{T} \text{ is a topology on } X \quad \text{and} \quad \mathcal{S} \subseteq \mathcal{T}\}.$$

The intersection over this family $\mathcal{F}_{\mathcal{S}}$ is a topology $\mathcal{T}(\mathcal{S})$ on X, and it is known as the topology generated by $\mathcal{S}$. It is the smallest topology on X in which every member of $\mathcal{S}$ is open.

Example 5.5. For any $\mathcal{S} \subseteq \{\emptyset, X\}$ in $\mathcal{P}(X)$,

$$\mathcal{T}(\mathcal{S}) \quad \text{is} \quad \text{the trivial topology and it is denoted by } \{\emptyset, X\}.$$

Example 5.6. For

$$\mathcal{S} = \{\text{all singletons } \{x\} \in \mathcal{P}(X) : x \in X\},$$

we note that

$$\mathcal{T}(\mathcal{S}) \quad \text{is} \quad \text{the discrete topology } \mathcal{P}(X).$$

Proposition 5.1 (basis for a topology). Let $\mathcal{B}$ be a collection of subsets of X such that the following hold:

(i) For each $x \in X$, there is at least one $B \in \mathcal{B}$ such that $x \in B$

(ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then

$$\text{there exists } B_3 \in \mathcal{B} \quad \text{such that} \quad x \in B_3 \subseteq B_1 \cap B_2.$$

Then, the topology $\mathcal{T}(\mathcal{B})$ generated by $\mathcal{B}$ is as follows: a subset U of X is open if and only if

$$\text{for every } x \in U \text{ there exists } B \in \mathcal{B} \quad \text{such that} \quad x \in B \text{ and } B \subseteq U.$$

We then say that $\mathcal{B}$ is a basis for the topology $\mathcal{T}(\mathcal{B})$.

Proof. Let $\mathcal{T}_0$ denote the set of all subsets $U \subseteq X$ such that for each $x \in U$, there exists $B \in \mathcal{B}$ with $x \in B$ and $B \subseteq U$. Then, given $U \in \mathcal{T}_0$, choose $B = B_x \in \mathcal{B}$, so

$$U = \bigcup_{x \in U} B_x.$$

Hence, if $\mathcal{T}$ is any topology on X with $\mathcal{B} \subseteq \mathcal{T}$, then $\mathcal{T}_0 \subseteq \mathcal{T}$, which shows that $\mathcal{T}_0 \subseteq \mathcal{T}(\mathcal{B})$.

We then prove the reverse inclusion, i.e. $\mathcal{T}_0 \supseteq \mathcal{T}(\mathcal{B})$. It suffices to show that $\mathcal{T}_0$ is a topology on X and $\mathcal{B} \subseteq \mathcal{T}_0$. Suppose $\{U_\alpha\}_{\alpha \in J}$ is a family of elements in $\mathcal{T}_0$ and

$$U = \bigcup_{\alpha \in J} U_\alpha.$$

Given $x \in U$, there exists an index $\alpha \in J$ such that $x \in U_\alpha$. Since $U_\alpha \in \mathcal{T}_\alpha$, then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U_\alpha$ which is $\subseteq U$. Hence, $U \in \mathcal{T}_0$. By **(i)** of Proposition 5.1, for all $x \in X$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq X$, SO $X \in \mathcal{T}_0$.

Next, suppose $U_1, U_2 \in \mathcal{T}_0$. So,

$$\begin{aligned} \text{given } x \in U_1 \cap U_2 \quad & \text{choose} \quad B_1 \in \mathcal{B} \text{ such that } x \in B_1 \subseteq U_1 \\ & \text{choose} \quad B_2 \in \mathcal{B} \text{ such that } x \in B_2 \subseteq U_2 \end{aligned}$$

By **(ii)** of Proposition 5.1,

$$\text{there exists } B_3 \in \mathcal{B} \quad \text{such that} \quad x \in B_3 \subseteq B_1 \cap B_2 \text{ which is } \subseteq U_1 \cap U_2.$$

Hence, $U_1 \cap U_2 \in \mathcal{T}_0$. By induction, for any finite family $\{U_\alpha\}_{\alpha \in J}$ of elements of $\mathcal{T}_0$, we have

$$\bigcap_{\alpha \in J} U_\alpha \in \mathcal{T}_0.$$

By **(iii)** of Definition 5.1, it follows that $\mathcal{T}_0$ is a topology on X. As such, given $B \in \mathcal{B}$, it is clear that for any $x \in B$, one has $x \in B \subseteq B$, so $B \in \mathcal{T}_0$. We conclude that $\mathcal{B} \subseteq \mathcal{T}_0$, so $\mathcal{T}_0 \supseteq \mathcal{T}(\mathcal{B})$. $\square$

Corollary 5.1. Let X be a set and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then,

$$\mathcal{T} \quad \text{is} \quad \text{equal to the collection of all unions of elements of } \mathcal{B}.$$

Definition 5.3 (metric topology)**.** Let (X,d) be a metric space. The metric topology of X with respect to the metric d is

$$\mathcal{T}_d = \{U_i \in \mathcal{P}(X) : \text{for all } p \in U \text{ there exists } r > 0 \text{ such that } B(p,r) \subseteq U\}.$$

That is to say, a subset $U \subseteq X$ is open with respect to the metric topology of X if and only if for all $p \in U$, there exists $r > 0$ such that $B(p,r) \subseteq U$.

Proposition 5.2. The metric topology (Definition 5.3) is indeed a topology on the metric space X.

Proof. It is clear that $\emptyset, X \in \mathcal{T}_d$. Next, suppose $\{U_i\}_{i\in I}$ is any collection in $\mathcal{T}_d$. Define

$$U = \bigcup_{i\in I} U_i.$$

We wish to prove that $U \in \mathcal{T}_d$. If $p \in U$, then there exists $i \in I$ such that $p \in U_i$ is open, so there exists $r > 0$ such that $B(p,r) \subseteq U_i \subseteq U$. It follows that $U \in \mathcal{T}_d$ (i.e. arbitrary union is contained in the topology).

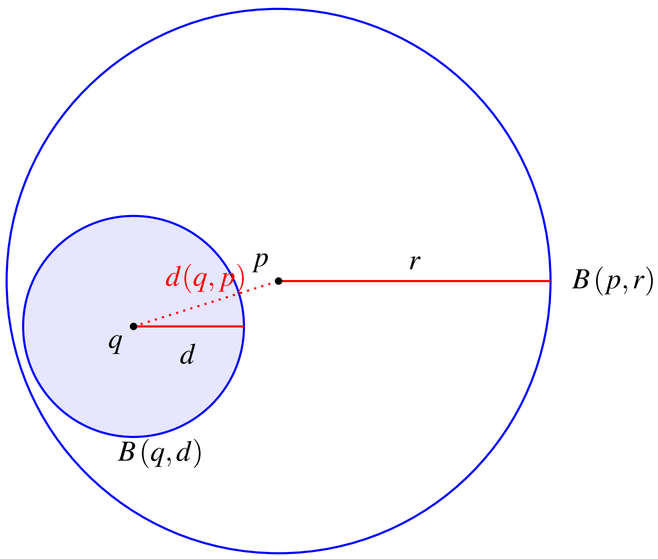
Then, suppose $\{U_i\}_{1\le i\le n}$ is a finite collection in $\mathcal{T}_d$. We wish to show that this finite union is also contained in the topology $\mathcal{T}_d$. Define

$$U = \bigcap_{i=1}^{n} U_i.$$

If $p \in U$, then for all $1 \le i \le n$, one has $p \in U_i$ being open in the metric topology, so there exists $r_i > 0$ such that $B(p,r_i) \subseteq U_i$. Define $r = \min\{r_i : 1 \le i \le n\} > 0$ since I is finite. Then, for all $i \in I$, one has $B(p,r) \subseteq U_i$ so $B(p,r) \subseteq U$. It follows that $U \in \mathcal{T}_d$. $\square$

Lemma 5.1. For all $p \in X$ and $r > 0$, the open ball $B(p,r)$ is indeed open with respect to the metric topology.

Proof. We wish to show that for every $q \in B(p,r)$, there exists $d > 0$ such that $B(q,d) \subseteq B(p,r)$. Since $q \in B(p,r)$, then $d(q,p) < r$ so $d = r - d(q,p) > 0$. We claim that for this $d > 0$, we indeed have $B(q,d) \subseteq B(p,r)$.



To see why the above claim holds, let $x \in B(q,d)$ so $d(x,q) < d$. By the triangle inequality, we have

$$d(x,p) \le d(x,q) + d(q,p) < d + (r-d) = r.$$

By the hypothesis, we mentioned that $d(x,q) < d$, and by definition, $d(q,p) = r - d$ so indeed, $d(x,p) < r$. We conclude that $x \in B(p,r)$. $\square$

Example 5.7 (basis for metric topology). Let (X,d) be a metric space (here, X is a set and d is a metric). A basis for the metric topology of X with respect to d is

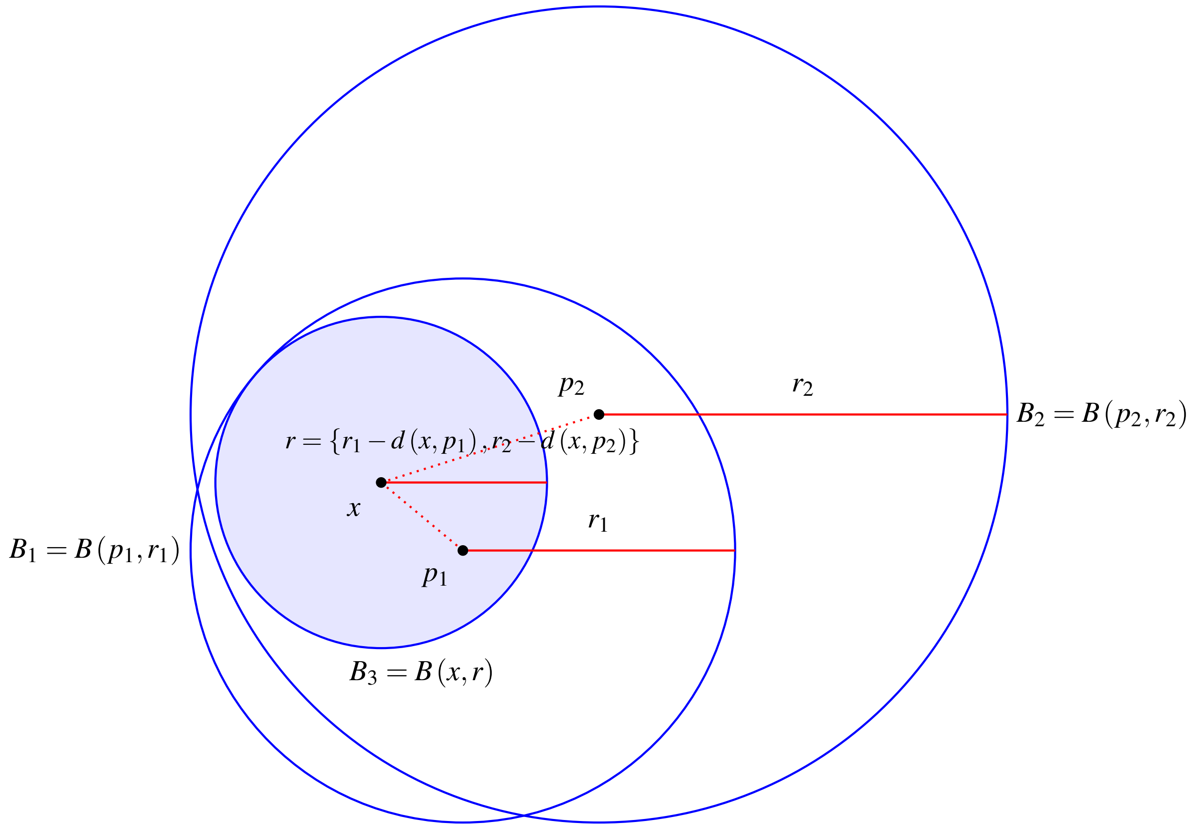
$$\mathcal{B} = \{\text{all open balls in } X\} = \{B(p,r) \in \mathcal{P}(X) : p \in X, r > 0\}.$$

The notation $B(p,r)$ here refers to an open ball of radius r centred at p.

We now relate to statements **(i)** and **(ii)** in Proposition 5.1. For **(i)**, it is clear that for all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$. Namely, we can choose $B = B(x,1)$. In fact, the number 1 can be replaced by any positive number.

As for **(ii)**, we need to show that if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Here, we can take $B_1 = B(p_1,r_1)$ and $B_2 = B(p_2,r_2)$. Then, we can set

$$r = \min\{r_1 - d(x,p_1), r_2 - d(x,p_2)\} > 0 \quad \text{and} \quad B_3 = B(x,r).$$



If $y \in B_3$, then for all $i = 1,2$, by the triangle inequality, we have

$$d(y,p_i) \le d(y,x) + d(x,p_i) < r + d(x,p_i) \le r_i.$$

As such, $x \in B_3 \subseteq B_1 \cap B_2$.

From Example 5.7, we say that a subset $U \subseteq X$ is open with respect to the metric topology of d if and only if for each $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$. Equivalently, U can be written as a union of open balls.

5.2 The Topological Notion of Continuity

Definition 5.4 (continuity). Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Proposition 5.3. Let X be a topological space. The identity map $\mathrm{id}_X : X \to X$ is continuous.

Proof. For any open $V \subseteq X$, note that $\mathrm{id}_X^{-1}(V) = V$ is open in X, so id_X is continuous. □

Proposition 5.4. Let X, Y, Z be topological spaces, if

$$\text{if } f : X \to Y \text{ and } g : Y \to Z \text{ are continuous} \quad \text{then} \quad g \circ f : X \to Z \text{ is continuous.}$$

Proof. For any open $W \subseteq Z$, we have $(g \circ f)^{-1}(W) = f^{-1}\left(g^{-1}(W)\right)$. Note that $g^{-1}(W)$ is open in Y by continuity of g; $f^{-1}(\cdot)$ is open in X by continuity of f. We conclude that $g \circ f$ is also continuous. □

Example 5.8 (trivial topology). Suppose the topology of Y is trivial, i.e. $\mathcal{T}_Y = \{\emptyset, Y\}$. Then, for any topological space X, any map $f : X \to Y$ from X to Y is continuous. As a consequence,

$$\text{for any topological space } X \quad \text{the unique map } X \to \{\cdot\} \text{ to a singleton} \quad \text{is continuous.}$$

This is because we can set $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$.

Example 5.9 (discrete topology). Suppose the topology of X is discrete, i.e. $\mathcal{T}_X = \mathcal{P}(X)$. Then, for any topological space Y, any map $f : X \to Y$ from X to Y is continuous because for any open $V \subseteq Y$, $f^{-1}(V) \subseteq X$. This implies that $f^{-1}(V)$ is open in X. As a consequence,

$$\text{for any topological space } Y \quad \text{any map } \{\cdot\} \to Y \text{ from a singleton} \quad \text{is continuous.}$$

That is to say, the unique map $\emptyset \to Y$ from the empty set is continuous.

Proposition 5.5. Suppose (X, d_X) and (Y, d_Y) are metric spaces given with the metric topology. Then, by the open set criterion for continuity,

$$\text{a map } f : X \to Y \quad \text{is continuous with respect to the metric topologies } d_X \text{ and } d_Y \quad \text{if and only if}$$
$$\text{it is continuous with respect to the metrics } d_X \text{ and } d_Y$$

Proof. For the reverse direction, suppose $f : X \to Y$ is continuous with repsect to the metrics d_X and d_Y, i.e. for all $p \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{for all } x \in X \text{ with } d_X(x, p) < \delta \quad \text{we have} \quad d_Y(f(x), f(p)) < \varepsilon.$$

We wish to show that for any open set $V \subseteq Y$, the set $f^{-1}(V) \subseteq X$ is open. Let $V \subseteq Y$ be an open set in Y. Let $p \in f^{-1}(V)$ be arbitrary, so $f(p) \in V$. Since $V \subseteq Y$ is open, then there exists $\varepsilon > 0$ such that $B(F(p), \varepsilon) \subseteq V$. By continuity with respect to the metrics d_X and d_Y, it implies that

$$f(B_X(p, \delta)) \subseteq B_Y(f(p), \varepsilon) \quad \text{which is } \subseteq V.$$

That is to say, $B_X(p, \delta) \subseteq f^{-1}(V)$ as desired.

As for the forward direction, suppose $f : X \to Y$ is continuous with respect to the metric topologies d_X and d_Y, i.e.

$$\text{for any open } V \subseteq Y \quad \text{the set } f^{-1}(V) \subseteq X \text{ is open.}$$

We wish to show that for any $p \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$ with $d_X(x, p) < \delta$, one has $d_Y(f(x), f(p)) < \varepsilon$. Let $p \in X$ and $\varepsilon > 0$ be given. Then, $f(p) \in Y$ and $V = B_Y(f(p), \varepsilon)$ is open in Y. As mentioned, by the open set crtierion for continuity, the set $f^{-1}(V) \subseteq X$ is open and $p \in f^{-1}(V)$ by definition of V.

Hence, there exists $\delta > 0$ such that $B_X(p, s) \subseteq f^{-1}(V)$. That is to say, $f(B_X(p, \delta)) \subseteq V$, i.e. for any $x \in X$ with $d_X(x, p) < \delta$, one has $f(x) \in V = B_Y(f(p), \varepsilon)$. In other words, $d_Y(f(x), f(p)) < \varepsilon$, and the result follows. □

Example 5.10 (Munkres p. 112 Question 2). Let $F : X \times Y \to Z$. We say that F is *continuous in each variable separately* if

$$\begin{aligned} &\text{for each } y_0 \in Y \quad \text{the map } h : X \to Z \text{ defined by } h(x) = F(x, y_0) \text{ is continuous} \quad \text{and} \\ &\text{for each } x_0 \in X \quad \text{the map } k : Y \to Z \text{ defined by } k(y) = F(x_0, y) \text{ is continuous} \end{aligned}$$

Show that if F is continuous, then F is continuous in each variable separately.

Solution. We first show continuity in the x-variable. Fix $y_0 \in Y$. Consider the map $h(x) = F(x, y_0)$. Take an arbitrary point x_0 in X. We must show that h is continuous at x_0. Let $U \subseteq Z$ be an open set containing $h(x_0) = F(x_0, y_0)$. Since F is continuous, for the open set U containing $F(x_0, y_0)$, there exists an open neighbourhood $V \subset X \times Y$ of (x_0, y_0) such that $F(V) \subseteq U$.

As such, there exist open sets $A \subseteq X$ and $B \subseteq Y$ with $x_0 \in A$ and $y_0 \in B$ such that $A \times B \subseteq Y$. Since $y_0 \in B$, for any $x \in A$ we have $(x, y_0) \in A \times B \subseteq V$, so $F(x, y_0) = h(x) \in U$. This shows that h is continuous at x_0. Similarly, one can prove continuity in the y-variable. □

Example 5.11 (Munkres p. 112 Question 12). Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation

$$F(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Show that F is continuous in each variable separately.
(b) Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x, x)$.
(c) Show that F is not continuous.

Solution.

(a) Recall the definition in Example 5.10. As F is symmetric, it suffices to show that for each $y_0 \in \mathbb{R}$,

$$h : \mathbb{R} \to \mathbb{R} \quad \text{defined by} \quad h(x) = \frac{xy_0}{x^2 + y_0^2} \text{ is continuous.}$$

Since $y_0 \neq 0$, then h can be regarded as a quotient of polynomials (i.e. h is a rational function). Hence, h is a continuous function.

(b) $g(x) = \frac{x^2}{2x^2} = \frac{1}{2}$.

(c) We have

$$\lim_{(x,x) \to (0,0)} \frac{x \cdot x}{x^2 + x^2} = \frac{1}{2} \quad \text{but} \quad \lim_{(x,0) \to (0,0)} \frac{x \cdot 0}{x^2 + 0^2} = 0$$

so F is not continuous at $(0, 0)$. □

5.3 Homeomorphisms

Definition 5.5 (homeomorphism)**.** Let X and Y be topological spaces. A continuous map $f : X \to Y$ is a homeomorphism if and only if

$$\text{there exists a continuous map } g : Y \to X \quad \text{such that} \quad g \circ f = \mathrm{id}_X \text{ and } f \circ g = \mathrm{id}_Y .$$

If g exists, it is unique and called the inverse map of f, denoted by f^{-1}. The topological spaces X and Y are homeomorphic if and only if there exists a homeomorphism $f : X \to Y$.

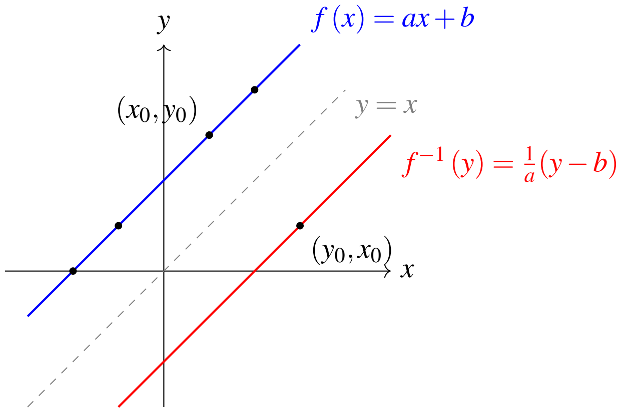
Properties of topological spaces which are invariant under homeomorphisms are called topological spaces.

Example 5.12. Let $X = Y = \mathbb{R}$, $a \in \mathbb{R} \setminus \{0\}$, and $b \in \mathbb{R}$. Then,

$$f : X \to Y \text{ given by } f(x) = ax + b \quad \text{is} \quad \text{a homeomorphism.}$$

The inverse map

$$f^{-1} : Y \to X \quad \text{is given by} \quad f^{-1}(y) = \frac{1}{a}(y - b).$$



Similarly, if $X = [0, 1]$ and $Y = [b, a + b]$, where $a > 0$, X and Y are homeomorphic.

Example 5.13. Let $X = (-1, 1)$ and $Y = \mathbb{R}$. Then,

$$f : X \to Y \quad \text{given by} \quad f(x) = \frac{x}{1 - |x|} \quad \text{is} \quad \text{a homeomorphism.}$$

The inverse map

$$f^{-1} : Y \to X \quad \text{is given by} \quad f^{-1}(y) = \frac{y}{1 - |y|}.$$

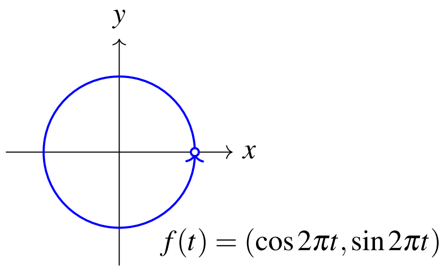
So, $(-1, 1)$ and $\mathbb{R}$ are homeomorphic.

At this juncture, we know that a homeomorphism is bijective, so cardinality is a topological property. However, there are bijective continuous maps which are not homeomorphisms! For example, let $\mathbb{S}^1$ denote the unit circle in $\mathbb{R}^2$. That is,

$$\mathbb{S}^1 = \left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \right\}.$$

Then, define

$$f : [0, 1) \to \mathbb{S}^1 \quad \text{where} \quad f(t) = (\cos 2\pi t, \sin 2\pi t).$$



The topology of $[0,1)$ is very different from that of $\mathbb{S}^1$ — $\mathbb{S}^1$ is compact (Definition 5.11) and connected (Definition 5.8) but $[0,1)$ is not compact! Alternatively, one can consider the discontinuity of the inverse function $f^{-1} : \mathbb{S}^1 \to [0,1)$ which tries to *unwrap* the circle into a line segment. But imagine what happens near the point $f(0) = (1,0)$. On the circle, if one goes slightly clockwise or slightly counter-clockwise from $(1,0)$, the t-values near those points are near 0 and 1 respectively. However, 1 is not in the domain! So, f^{-1} has a discontinuity at $(1,0)$. Therefore, the inverse is not continuous, which implies f is not a homeomorphism.

Example 5.14 (Munkres p. 157 Question 1). Show that no two of the spaces $(0,1)$, $(0,1]$, and $[0,1]$ are homeomorphic.
Hint: What happens if you remove a point from each of these spaces?

Solution. We can distinguish these spaces by examining their cut-points. These refer to points whose removal disconnects the space. This is a topological property preserved by homeomorphisms.

For $(0,1)$, take $x \in (0,1)$. Removing x from $(0,1)$ yields

$$(0,1) \setminus \{x\} = (0,x) \cup (x,1).$$

Both $(0,x)$ and $(x,1)$ are open and non-empty, and they are disjoint. Hence, the space is disconnected. In other words, every point of $(0,1)$ is a cut point.

Next, for $(0,1]$, consider the point 1. Removing 1 yields $(0,1] \setminus \{1\} = (0,1)$, which as an interval in $\mathbb{R}$ is connected. So, 1 is not a cut-point. As such, in $(0,1]$, at least one point is not a cut-point.

Lastly, we deal with the closed interval $[0,1]$. Consider one of the endpoints, say 0. Removing 0 yields $[0,1] \setminus \{0\} = (0,1]$ which is still connected. So, 0 is not a cut-point. The same holds for the point 1.

In conclusion, in $(0,1)$, every point is a cut-point; in $(0,1]$, there is at least one point that is not a cut-point; in $[0,1]$, there are at least two points that are not cut-points. Since the property of 'every point being a cut point' is a topological invariant (it must be preserved by any homeomorphism), no homeomorphism can exist between $(0,1)$ and either $(0,1]$ or $[0,1]$. Similarly, because the number of non-cut points differs, $(0,1]$ and $[0,1]$ cannot be homeomorphic. The result follows. $\square$

Example 5.15 (Munkres p. 157 Question 1). Show $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

Solution. If we remove a single point $x \in \mathbb{R}^n$, then the remaining set is still connected. However, if we remove a single point $x \in \mathbb{R}$, then the remaining set is disconnected. $\square$

Example 5.16 (Munkres p. 158 Question 2). Let $f : \mathbb{S}^1 \to \mathbb{R}$ be a continuous map. Show there exists a point x of $\mathbb{S}^1$ such that $f(x) = f(-x)$[†].

[†]This is the Borsuk-Ulsam theorem for $n = 1$.

Solution. Here, $\mathbb{S}^1$ denotes the unit circle in $\mathbb{R}^2$. Define $g(x) = f(x) - f(-x)$. Then, $g(-x) = -g(x)$ so g is an odd function on the circle.

We then parametrise $\mathbb{S}^1$ using θ so that every point on the circle can be written as

$$x(\theta) = (\cos\theta, \sin\theta) \quad \text{where } 0 \le \theta \le 2\pi.$$

In particular, $x(\theta + \pi) = -x(\theta)$. Define $h(\theta) = g(x(\theta))$. Since h is a composition of continuous functions, then it is also continuous. We see that

$$h(0) = g(x(0)) = f(x(0)) - f(x(\pi))$$

and

$$h(\pi) = f(x(\pi)) - f(x(2\pi)) = f(x(\pi)) - f(x(0)) = -h(0).$$

Since $h(0)$ and $h(\pi)$ are negatives of each other, either $h(0) = 0$ (for which the result follows) or $h(0) = h(\pi)$. For the latter case, by the intermediate value theorem, there exists $\theta_0 \in (0, \pi)$ such that $h(\theta_0) = 0$. As such, $f(x(\theta_0)) = f(-x(\theta_0))$. That is to say, there exists a point $x \in \mathbb{S}^1$ such that $f(x) = f(-x)$. □

Example 5.17 (Munkres p. 158 Question 3)**.** Let $f : X \to X$ be continuous. Show that if $X = [0,1]$, there is a point x such that $f(x) = x$.

Solution. Define $g(x) = f(x) - x$. Then, g is continuous. Since $0 \le f(0), f(1) \le 1$, then $g(0) \ge 0$ and $g(1) \le 0$, so by the intermediate value theorem, there exists $k \in (0,1)$ such that $g(k) = 0$, i.e. $f(k) = k$. □

5.4 The Subspace Topology

Definition 5.6 (subspace topology)**.** Let X be a topological space with topology $\mathcal{T}$. If $Y \subseteq X$,

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\} \quad \text{is} \quad \text{a topology on } Y$$

and it is called the subspace topology of Y induced by X.

We shall prove that the subsapce topology is indeed a topology.

Proof. We have

$$(U_1 \cap Y) \cap \ldots (U_n \cap Y) = (U_1 \cap \ldots \cap U_n) \cap Y$$

so the finite intersection $U_1 \cap \ldots \cap U_n$ is contained in the topology. Moreover, the arbitrary union

$$\bigcup_{\alpha \in J} (U_\alpha \cap Y) = \left(\bigcup_{\alpha \in J} U_\alpha \right) \cap Y$$

is contained in the topology as well. □

Unless otherwise stated, any subset of a topological space is given the subspace topology.

Lemma 5.2. If A is a subspace of X,

$$\text{the inclusion function } j : A \to X \quad \text{is} \quad \text{continuous.}$$

Proof. For any open $U \subseteq X$, one has $j^{-1}(U) = U \cap A$. By the definition of the subspace topology (Definition 5.6), $U \cap A$ is open in A. $\square$

Example 5.18. If $f : X \to Y$ is continuous and if $A \subseteq X$ is a subspace of X, then the restriction map $f|_A : A \to Y$ is continuous because

$$\begin{tikzcd} A \arrow[rr, "f|_A"] \arrow[dr, hook, "j"'] & & Y \\ & X \arrow[ur, "f"'] & \end{tikzcd} \qquad \text{commutes.}$$

Here, $j : A \hookrightarrow X$ is the inclusion map, which is continuous by the definition of the subspace topology. Since $f = f|_A \circ j$, and both f and j are continuous, it follows that $f|_A$ is continuous.

Example 5.19. Let $f : X \to Y$ be continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous because

$$\begin{tikzcd} & Y \arrow[dr, hook, "j"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd} \qquad \text{commutes.}$$

Proposition 5.6 (universal property of the subspace topology)**.** Let X be a subspace topology and let $A \subseteq X$ be a subset given with the subspace topology. For any topological space T and any map $f_0 : T \to A$,

$$f_0 : T \to A \text{ is continuous} \quad \text{if and only if} \quad j \circ f_0 : T \to X \text{ is continuous.}$$

Proof. Since A is given the subspace topology and j is continuous, then f_0 is continuous. So, $j \circ f_0$ is continuous as the composition of continuous maps is also continuous.

$$\begin{tikzcd} & A \arrow[r, hook, "j"] & X \\ T \arrow[ur, "f_0"] \arrow[urr, bend right, "j\circ f_0"'] & & \end{tikzcd}$$

Conversely, if $j \circ f_0$ is continuous, then by definition of the subspace topology of A,

$$\text{for all open } V \subseteq A \text{ there exists open } U \subseteq X \quad \text{such that} \quad V = A \cap U = j^{-1}(U).$$

Hence,

$$f_0^{-1}(V) = f_0^{-1}\left(j^{-1}(U)\right) = f^{-1}(U) \quad \text{is} \quad \text{open in } T,$$

so we conclude that f_0 is continuous. $\square$

5.5
Connectedness

Definition 5.7 (separation)**.** Let X be a topological space. A separation of X is a pair U, V of disjoint non-empty open subsets of X whose union is X.

Definition 5.8 (connectedness). Let X be a topological space. Then, the space X is said to be connected if and only if there does not exist a separation of X. That is to say,

$$\text{for all open } U,V \text{ of } X \quad \text{such that} \quad U \cap V = \emptyset \text{ and } U \cup V = X \quad \text{either} \quad U = \emptyset \text{ or } V = \emptyset.$$

In other words, the only subsets of X that are both open and closed in X are $\emptyset$ and X.

Example 5.20. Suppose the topology of X is trivial. Then, X is connected implies that any topological space X such that X is empty or a singleton is connected.

Example 5.21. Suppose the topology of X is discrete. Then,

$$X \text{ is connected} \quad \text{if and only if} \quad X \text{ is empty or a singleton.}$$

From Examples 5.20 and 5.21, we see that connectedness for a topological space X generalises the notion of 'being empty or a singleton' for a set.

Also, note that subspaces of a connected space need not be connected (Examples 5.22 and 5.23).

Example 5.22. For example, $\mathbb{R} \setminus \{0\}$ with the subspace topology is not connected. To see why, take $(-\infty, 0)$ and $(0, \infty)$ to be disjoint non-empty open sets whose union is $\mathbb{R} \setminus \{0\}$.

Example 5.23. Also, $\mathbb{Q} \subseteq \mathbb{R}$ with the subspace topology is not connected. To see why, for any irrational $\alpha \in \mathbb{R} \setminus \mathbb{Q}$,

$$(-\infty, \alpha) \cap \mathbb{Q} \text{ and } (\alpha, \infty) \cap \mathbb{Q} \quad \text{are} \quad \text{disjoint non-empty open sets whose union is } \mathbb{Q}.$$

Theorem 5.1 (image of connected space under continuous map is connected). If $f : X \to Y$ is a continuous map and X is connected, then $f(X)$ given with the subspace topology from Y is connected.

Proof. Consider the map

$$f_0 : X \twoheadrightarrow f(X) \quad \text{where} \quad f_0(x) = f(x).$$

Here, $\twoheadrightarrow$ means that f_0 is surjective. By the universal property of the subspace topology (Proposition 5.6), f_0 is continuous and surjective so one may replace Y and f by $f(X)$ and f_0 respectively.

$$\begin{tikzcd} & f(X) \arrow[r, hook, "i"] & Y \\ X \arrow[ur, "f_0"] \arrow[urr, bend right, "f"'] & & \end{tikzcd}$$

Without loss of generality, one may assume that $f : X \twoheadrightarrow Y$ is continuous and surjective. So, it suffices to show that X connected implies Y connected. We shall prove this result.

Let U and V be open subsets of Y such that

$$U \cap V = \emptyset \quad \text{and} \quad U \cup V = Y.$$

We wish to prove that one of U, V is empty. Since f is continuous, then $f^{-1}(U)$ and $f^{-1}(V)$ are open subsets of X and

$$f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = f^{-1}(Y) = X \quad \text{and} \quad f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset.$$

Since X is connected, either $f^{-1}(U) = \emptyset$ or $f^{-1}(V) = \emptyset$. Next, since f is surjective, then $U = f\left(f^{-1}(U)\right)$ or $V = f\left(f^{-1}(V)\right)$, where either U or V is empty. $\square$

Corollary 5.2 (connectedness is a topological property). If X and Y are homeomorphic topological spaces, then

$$X \text{ is connected} \quad \text{if and only if} \quad Y \text{ is connected.}$$

Theorem 5.2 (classification of connected subspaces of $\mathbb{R}$). Let $I \subseteq \mathbb{R}$ be any subset of $\mathbb{R}$. Then, the following are equivalent:

(i) I is connected

(ii) I is convex, i.e. for all $x, y \in I$ such that $x \leq y$, we have $[x, y] \subseteq I$

(iii) I is an interval, i.e. one of he following:

$$(a,b), (a,b], [a,b), [a,b], (-\infty,b), (-\infty,b], (a,\infty), [a,\infty) \quad \text{or} \quad \emptyset \quad \text{or} \quad \mathbb{R} = (-\infty,\infty).$$

Definition 5.9 (convexity). A subset of the Euclidean plane $E \subseteq \mathbb{R}^k$ is

$$\text{convex} \quad \text{if and only if} \quad \text{for any } \mathbf{x}, \mathbf{y} \in E \text{ one has } [\mathbf{x}, \mathbf{y}] \subseteq E.$$

Here, $[\mathbf{x}, \mathbf{y}] = \left\{\lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in \mathbb{R}^k : \lambda \in [0,1]\right\}$.

Proposition 5.7. Convex subsets of Euclidean spaces are connected.

5.6 Compactness

Definition 5.10 (cover and open cover). Let X be a topological space. A collection $\mathcal{A}$ of subsets of a space X is said to cover X or a covering of X if and only if the union of the elements of $\mathcal{A}$ is equal to X. That is to say

$$X = \bigcup_{U \in \mathcal{A}} U.$$

It is called an open covering if and only if its elements are open subsets of X.

Definition 5.11 (compactness). A topological space X is said to be compact if and only if every open covering $\mathcal{A}$ of X reduces to a finite subcollection that also covers X. That is to say, for any family $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of open subsets of X such that

$$X = \bigcup_{\alpha \in \mathcal{A}} U_\alpha,$$

there exists a finite subset $\mathcal{A}_0 \subseteq \mathcal{A}$ such that

$$X = \bigcup_{\alpha \in \mathcal{A}_0} U_\alpha = U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}.$$

Example 5.24. If the topology of X is trivial, then X is compact.

Example 5.25. If the topology of X is discrete, then

$$X \text{ is compact} \quad \text{if and only if} \quad X \text{ is finite.}$$

To see why, for the forward direction, suppose X is compact and has the discrete topology. Recall Example 5.1, which states that every subset of X is open. In particular, for every $x \in X$, the singleton $\{x\}$ is open. Now, consider the open cover

$$\mathcal{A} = \{\{x\} : x \in X\}$$

which is an open cover of X because

$$\bigcup_{x \in X} \{x\} = X.$$

Since X is compact, there exists a finite subcover $\{\{x_1\}, \{x_2\}, \ldots, \{x_n\}\}$ such that

$$\bigcup_{i=1}^{n} \{x_i\} = X.$$

Hence, $X = \{x_1, \ldots, x_n\}$, implying that X is a finite set.

Next, suppose X is finite and has the discrete topology. Let $\mathcal{A}$ be any open cover of X. Since every subset of X is open and X has finitely many elements, we can write $X = \{x_1, \ldots, x_n\}$. For every x_i, there exists some $A_i \in \mathcal{A}$ such that $x_i \in A_i$. So, the finite subcollection $\{A_1, \ldots, A_n\} \subseteq \mathcal{A}$ covers X since every point is covered. Hence, every open cover has a finite subcover, and X is compact.

Example 5.26. Any finite topological space X is compact.

In essence, compactness for a topological space X generalises the notion of finiteness for a set.

We will see in due course that the compact subsets of $\mathbb{R}$ are the closed and bounded subsets of $\mathbb{R}$. That is, the closed and bounded interval $[a,b]$, where $a < b$, is compact. However, $\mathbb{R}$ is not compact. The reason is as follows. Consider the open cover $\{(-n,n)\}_{n \in \mathbb{N}}$ of $\mathbb{R}$, which does not reduce to a finite subcover.

Also, note that a subspace of a compact space need not be compact. For example, consider $(0,1]$ with the subspace topology from $[0,1]$. We claim that $(0,1]$ is not compact. Let $\left\{\left(\frac{1}{n}, 1\right]\right\}_{n \in \mathbb{N}}$ be an open cover of $(0,1]$, which does not reduce to a finite subcover.

Also, $\mathbb{Q} \cap [0,1]$ with subspace topology from $[0,1]$ is not compact. To see why, let α be an irrational contained in $[0,1]$. Then, $\left\{\mathbb{Q} \cap [0,1] \setminus \left[\alpha - \frac{1}{n}, \alpha + \frac{1}{n}\right]\right\}_{n \in \mathbb{N}}$ is an open cover of $\mathbb{Q} \cap [0,1]$ but it does not reduce to a finite subcover.

Theorem 5.3 (image of compact space under continuous map is compact)**.** If $f : X \to Y$ is a continuous map and X is compact, then $f(X)$ given with the subspace topology from Y is compact.

Proof. Consider the map

$$f_0 : X \twoheadrightarrow f(X) \quad \text{where} \quad f_0(x) = f(x).$$

By the universal property of the subspace topology (Proposition 5.6), f_0 is continuous and surjective so one may replace Y and f by $f(X)$ and f_0 respectively.

$$\begin{tikzcd}
 & f(X) \arrow[r, hook, "i"] & Y \\
X \arrow[ur, "f_0"] \arrow[urr, bend right, "f"'] & &
\end{tikzcd}$$

Without loss of generality, we may assume that $f : X \twoheadrightarrow Y$ is continuous and surjective, and show that X compact implies Y compact. To see why this holds, let $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of Y, so

$$Y = \bigcup_{\alpha\in\mathcal{A}} V_\alpha.$$

Then, $\left\{f^{-1}(V_\alpha)\right\}_{\alpha\in\mathcal{A}}$ is an open cover of X because f is continuous, i.e.

$$X = \bigcup_{\alpha\in\mathcal{A}} f^{-1}(V_\alpha) = f^{-1}\left(\bigcup_{\alpha\in\mathcal{A}} V_\alpha\right).$$

Since X is compact, there exists a finite subset $\mathcal{A}_0 \subseteq \mathcal{A}$ such that

$$X = \bigcup_{\alpha\in\mathcal{A}_0} f^{-1}(V_\alpha).$$

Since f is surjective, then

$$Y = \bigcup_{\alpha\in\mathcal{A}_0} V_\alpha$$

which implies that Y admits a finite subcover as well. We conclude that Y is compact. □

Corollary 5.3 (compactness is a topological property)**.** If X and Y are homeomorphic topological spaces, then

$$X \text{ is compact} \quad \text{if and only if} \quad Y \text{ is compact.}$$

Example 5.27 (Munkres p. 171 Question 3)**.** Show that a finite union of compact subsets of X is compact.

Solution. Let $Y_1,\ldots,Y_n$ be compact subsets of X. That is to say, for every $1 \le i \le n$ and every collection $\mathcal{C}$ of open sets covering Y_i, i.e.

$$Y_i = \bigcup_{S\in\mathcal{C}} S,$$

there exists a finite subcollection $\mathcal{F}_i \subseteq \mathcal{C}$ such that

$$Y_i = \bigcup_{S\in\mathcal{F}_i} S.$$

We wish to show that $\bigcup_{i=1}^n Y_i$ is compact. Consider an arbitrary open cover of the aforementioned union. Since $\mathcal{C}$ covers $\bigcup_{i=1}^n Y_i$, it covers each Y_i individually. Define

$$\mathcal{F} = \bigcup_{i=1}^n \mathcal{F}_i.$$

Since there are finitely many i and each $\mathcal{F}_i$ is a finite set, then $\mathcal{F}$ is a finite collection of open sets. It follows that $\mathcal{F}$ is a finite subcover of $\bigcup_{i=1}^n Y_i$. □

Example 5.28 (Bartle and Sherbert p. 337 Question 7)**.** Find an infinite collection $\{K_n : n \in \mathbb{N}\}$ of compact sets in $\mathbb{R}$ such that

$$\text{the union } \bigcup_{n=1}^\infty K_n \quad \text{is not compact.}$$

Solution. Define $K_n = [-n,n]$, where $n \in \mathbb{N}$. Then, K_n is compact for all $n \in \mathbb{N}$. However,

$$\bigcup_{n=1}^\infty K_n = \lim_{N\to\infty} \bigcup_{n=1}^N [-n,n] = \mathbb{R}$$

which is not compact. □

Example 5.29 (Bartle and Sherbert p. 337 Question 9). Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of non-empty compact sets in $\mathbb{R}$ such that

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots.$$

Prove that there exists at least one point $x \in \mathbb{R}$ such that $x \in K_n$ for all $n \in \mathbb{N}$; that is,

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

Solution. For each $n \in \mathbb{N}$, choose $x_n \in K_n$. Then, $x_n \in K_1$. Since K_1 is compact, by the Bolzano-Weierstrass theorem, there exists a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ that converges to some limit $x \in K_1$. Fix any $m \in \mathbb{N}$. For large k (i.e. such that $n_k \geq m$), we have

$$x_{n_k} \in K_{n_k} \subseteq K_m.$$

As K_m is compact (thus closed by the Heine-Borel theorem), it contains the limit of every convergent sequence of its points. Hence, the limit x must lie in K_m. Since m was arbitrary, then $x \in K_m$ for every $m \in \mathbb{N}$. That is to say

$$x \in \bigcap_{m=1}^{\infty} K_m.$$

Therefore, the intersection

$$\bigcap_{n=1}^{\infty} K_n$$

is non-empty. □

Example 5.30 (Munkres p. 171 Question 4). Show that every compact subspace of a metric space is bounded in that metric and is closed.

Solution. Let (X,d) be a metric space, and let $K \subseteq X$ be a compact subspace. We wish to prove that K is bounded and closed.

We first prove that K is bounded. That is, we need to find some point $x_0 \in X$ and $R > 0$ such that

$$K \subseteq B(x_0,R) = \{x \in X : d(x,x_0) < R\}.$$

Pick any point $x_0 \in X$. Consider the collection of open balls

$$\{B(x_0,n) : n \in \mathbb{N}\},$$

which form an open cover of X, and hence of K. Since K is compact, there exists a finite subcover

$$K \subseteq \bigcup_{i=1}^{m} B(x_0,n_i).$$

Let $R = \max\{n_1,\ldots,n_m\}$. Then $K \subseteq B(x_0,R)$, showing that K is bounded.

To show that K is closed, it suffices to show that $X \setminus K$ is open. Let $x \in X \setminus K$. Note that K is compact and X is a metric space and hence Hausdorff. Recall that compact sets in Hausdorff spaces are closed. For each $y \in K$, since $x \neq y$, we have $d(x,y) > 0$. The function $d(x,\cdot) : K \to \mathbb{R}$ is continuous, and since K is compact, the function achieves a minimum distance, denoted by

$$\delta = \inf\{d(x,y) : y \in K\} > 0.$$

Then $B\left(x, \frac{\delta}{2}\right) \subseteq X \setminus K$, so every point of $X \setminus K$ is an interior point, and $X \setminus K$ is open. Thus, K is closed. □

Definition 5.12. A subset Y of a topological space X is compact if and only if Y given with the subspace topology is compact.

Theorem 5.4. Every closed subspace of a compact space is compact.

5.7 Closed Sets

Definition 5.13 (closed set). A subset A of a topological space X is closed if the set $X \setminus A$ is open.

Theorem 5.5. Let X be a topological space. Then, the following conditions hold:

(i) $\emptyset$ and X are closed

(ii) Arbitrary intersections of closed sets are closed

(iii) Finite unions of closed sets are closed

In **(ii)** of Theorem 5.5, let U_α and C_α be an open set and a closed set of X respectively. Then,

$$\bigcap_{\alpha\in\mathcal{A}} C_\alpha = \bigcap_{\alpha\in\mathcal{A}} (X \setminus U_\alpha) = X \setminus \bigcup_{\alpha\in\mathcal{A}} U_\alpha,$$

where we used de Morgan's law in the second equality.

Example 5.31 (Bartle and Sherbert p. 332 Question 5). Show that the set $\mathbb{N}$ of natural numbers is a closed set in $\mathbb{R}$.

Solution. It suffices to show that $\mathbb{R} \setminus \mathbb{N}$ is an open set in $\mathbb{R}$. Let $a \in \mathbb{R} \setminus \mathbb{N}$ be arbitrary. Choose

$$\varepsilon = \frac{1}{2}\min\{\lceil a\rceil - a, a - \lfloor a\rfloor\} > 0.$$

Then, consider the ε-neighbourhood $(a-\varepsilon, a+\varepsilon)$ which is strictly *trapped* between two integers. As such $V_\varepsilon(a)$ is an open in $\mathbb{R}$ □

Example 5.32 (Bartle and Sherbert p. 332 Question 7). Show that the set $\mathbb{Q}$ of rational numbers is neither open nor closed in $\mathbb{R}$.

Solution. Suppose on the contrary that $\mathbb{Q}$ is open in $\mathbb{R}$. Then, there exists $\varepsilon > 0$ such that for any $a \in \mathbb{Q}$, $V_\varepsilon(a)$ only contains rational numbers. However, no such ε exists because the irrational numbers $\mathbb{Q}'$ are dense in $\mathbb{R}$. So, every open interval in $\mathbb{R}$ contains at least one irrational number, contradicting $V_\varepsilon(a) \subseteq \mathbb{Q}$. Hence, $\mathbb{Q}$ is not open in $\mathbb{R}$.

We then prove that $\mathbb{Q}$ is not closed in $\mathbb{R}$. Suppose on the contrary that $\mathbb{Q}$ is closed in $\mathbb{R}$, i.e. $\mathbb{R} \setminus \mathbb{Q}$ is open in $\mathbb{R}$. In a similar fashion, by using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, then every open interval of $\mathbb{R}$ contains some rational numbers. Hence, $\mathbb{R} \setminus \mathbb{Q}$ is not open in $\mathbb{R}$, so $\mathbb{Q}$ is not closed in $\mathbb{R}$. □

Example 5.33 (Bartle and Sherbert p. 332 Question 6). Show that

$$A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \text{ is not a closed set} \quad \text{but} \quad A \cup \{0\} \text{ is a closed set in } \mathbb{R}.$$

Solution. For the first part, we note that a set is closed in $\mathbb{R}$ if it contains all its limit points. As such, we shall prove that A has a limit point which is not an element of A. Let $x_n = \frac{1}{n}$ and consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ in A. As

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0,$$

then it implies that every open interval around 0 contains points of A. As such, 0 is a limit point of A. However, $0 \notin A$ because every element of A is of the form $\frac{1}{n}$, where $n \in \mathbb{N}$, so $\frac{1}{n} > 0$. Hence, A does not contain all its limit points, so A is not closed.

We then prove that $B = A \cup \{0\}$ is a closed set in $\mathbb{R}$. Previously, we mentioned that 0 is a limit point of A. We claim that no other point in $\mathbb{R}$ is a limit point of B. To see why, for any $\frac{1}{n} \in \mathbb{R}$, there exists $\varepsilon > 0$ such that

$$\left(\frac{1}{n} - \varepsilon, \frac{1}{n} + \varepsilon\right) \cap A = \left\{\frac{1}{n}\right\}.$$

This implies that for any $n \in \mathbb{N}$, $\frac{1}{n}$ is isolated in A. Since B contains all its limit points, then B is closed in $\mathbb{R}$. $\square$

Example 5.34 (Bartle and Sherbert p. 333 Question 18)**.** Show that if $F \subseteq \mathbb{R}$ is a closed non-empty set that is bounded above, then $\sup F \in F$.

Solution. Let $s = \sup F$. Since F is non-empty and bounded above, by the completeness of $\mathbb{R}$, s exists. For each $n \in \mathbb{N}$, consider the interval $\left(s - \frac{1}{n}, s\right]$. By definition of supremum, s is the least upper bound, so $s - \frac{1}{n}$ cannot be an upper bound for F. Hence, for every n, there exists an element $x_n \in F$ such that

$$s - \frac{1}{n} < x_n \leq s.$$

This produces a sequence $\{x_n\}_{n\in\mathbb{N}}$ in F. The inequality

$$0 \leq s - x_n < \frac{1}{n}$$

implies that

$$\lim_{n\to\infty} (s - x_n) = 0 \quad \text{so} \quad \lim_{n\to\infty} x_n = s.$$

Since F is closed, it contains all its limit points. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is contained in F and converges to s. Therefore, the limit s must belong to F. $\square$

Definition 5.14 (interior point)**.** A point $x \in \mathbb{R}$ is said to be an *interior point* of $A \subseteq \mathbb{R}$ in case there is a neighbourhood V of x such that $V \subseteq A$.

Definition 5.15 (boundary point)**.** A point $x \in \mathbb{R}$ is said to be a *boundary point* of $A \subseteq \mathbb{R}$ in case every neighbourhood V of x contains points in A and points in A'.

Example 5.35 (Bartle and Sherbert p. 332 Question 11)**.** Show that a set

$$G \subseteq \mathbb{R} \text{ is open} \quad \text{if and only if} \quad \text{it does not contain any of its boundary points.}$$

Solution. For the forward direction, suppose $G \subseteq \mathbb{R}$ is open. Since G is open, then for every $x \in G$, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq G$. Suppose on the contrary that there exists a point $x \in G$ that is also a boundary point of G. Then, every neighbourhood of x must intersect $\mathbb{R} \setminus G$. However, $(x - \varepsilon, x + \varepsilon)$ does not intersect G, contradicting the assumption that x is a boundary point.

For the reverse direction, suppose G does not contain any of its boundary points. That is, $G \cap \partial G = \emptyset$, where ∂G denotes the boundary of G. We wish to show that G is open. Suppose on the contrary that G is not open. Then, there exists at least one point $x \in G$ for which no open interval around x is entirely contained in G. That is to say, for every $\varepsilon > 0$,

$$(x-\varepsilon, x+\varepsilon) \quad \text{contains points that are not in } G.$$

Hence, $(x-\varepsilon, x+\varepsilon) \cap \mathbb{R} \setminus G \neq \emptyset$. It follows that x is a boundary point of G, i.e. $x \in \partial G$. This contradicts our assumption that G contains no boundary points, and the result follows. □

Similar to Example 5.35, one can show that a set $F \subseteq \mathbb{R}$ is closed if and only if it contains all of its boundary points.

Definition 5.16 (interior). Let X be a topological space. If $A \subseteq X$, let

$$\operatorname{Int}(A) \text{ or } A^\circ \quad \text{be} \quad \text{the union of all open sets that are contained in } A.$$

The set A° is called the *interior* of A.

Example 5.36 (Bartle and Sherbert p. 332 Question 13). If $A \subseteq \mathbb{R}$, show that A° is an open set, that it is the largest open set contained in A, and that a point z belongs to A° if and only if z is an interior point of A.

Solution. We first prove that A° is an open set, i.e. the union of all open sets that are contained in A is also open in A. Let $x \in A^\circ$. Then there exists an open set $U \subseteq A$ such that $x \in U$. Since U is open, there exists $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq U$. But $U \subseteq A^\circ$ (since A° is the union of all such open sets), hence $V_\varepsilon(x) \subseteq A^\circ$. This shows that every point $x \in A^\circ$ has an open neighbourhood in A°, so A° is open.

We then show that A° is the largest open set contained in A. Let $U \subseteq A$ be an arbitrary open set. Since A° by definition is the union of all open sets that are contained in A, then $U \subseteq A^\circ$, so A° contains every open set contained in A.

Lastly, we prove that

$$z \in A^\circ \quad \text{if and only if} \quad z \text{ is an interior point of } A.$$

Suppose $z \in A^\circ$. Then, $z \in A_i$, where $A_i \subseteq A$ is some open set. In fact, the forward and backward direction follow immediately from the definition of an interior point (Definition 5.14). □

Example 5.37 (Bartle and Sherbert p. 333 Question 14). Using the notation of Example 5.36, let A, B be sets in $\mathbb{R}$. Show that

$$A^\circ \subseteq A \quad \text{and} \quad (A^\circ)^\circ = A^\circ \quad \text{and} \quad (A \cap B)^\circ = A^\circ \cap B^\circ.$$

Show also that $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$, and give an example to show that the inclusion may be proper.

Solution. We first prove that $A^\circ \subseteq A$. Let $x \in A^\circ$ be arbitrary. Then, $x \in A_i$, where A_i is an open set contained in A. Hence, $x \in A$, so it follows that $A^\circ \subseteq A$.

We then prove that $(A^\circ)^\circ = A^\circ$. Let $x \in (A^\circ)^\circ$. Suppose $x \in (A^\circ)^\circ$. By definition of the interior, there exists an open set U such that

$$x \in U \quad \text{and} \quad U \subseteq A^\circ.$$

Since $U \subseteq A^\circ$, then $x \in A^\circ$, so the forward inclusion holds. As for the reverse inclusion, suppose $y \in A^\circ$. Then, there exists an open set V such that $y \in V \subseteq A^\circ$. So, $y \in (A^\circ)^\circ$, and the result follows.

We then prove that $(A\cap B)^\circ = A^\circ \cap B^\circ$. For the forward inclusion, suppose $x \in (A\cap B)^\circ$. Then, x is in the union of all open sets contained in $A\cap B$. In particular, there exists an open set U such that $x \in U \subseteq A\cap B$. Since $x \in A\cap B$, then $x \in A$ and $x \in B$. Since

$$x \in A \quad \text{then} \quad x \in \text{some open set contained in } A.$$

In particular, $x \in A^\circ$. Similarly, one can deduce that $x \in B^\circ$. It follows that $x \in A^\circ \cap B^\circ$. One can prove the reverse inclusion in a similar manner.

Lastly, we prove that $A^\circ \cup B^\circ \subseteq (A\cup B)^\circ$. Suppose $x \in A^\circ \cup B^\circ$. Then, $x \in A^\circ$ or $x \in B^\circ$. If $x \in A^\circ$, then $x \in U$ for some open set contained in A. Hence, $x \in A$. One can then show that $x \in A$ or $x \in B$, so $x \in A\cup B$. Hence, $x \in (A\cup B)^\circ$.

In fact, the inclusion can be proper. Let $A = [0,1]$ and $B = [1,2]$. Then,

$$A^\circ = (0,1) \text{ and } B^\circ = (1,2) \quad \text{so} \quad A^\circ \cup B^\circ = (0,1)\cup(1,2).$$

However,

$$A\cup B = [0,2] \quad \text{so} \quad (A\cup B)^\circ = (0,2).$$

This shows that $1 \in (A\cup B)^\circ$ but $1 \notin A^\circ \cup B^\circ$. □

Similar to Definition 5.16, we give the definition of the closure of a set (Definition 5.17).

Definition 5.17 (closure)**.** Let X be a topological space. If $A \subseteq X$, let

$$\mathrm{Cl}(A) \text{ or } A^- \text{ or } \overline{A} \quad \text{be} \quad \text{the intersection of all closed sets containing } A.$$

The set A^- is called the *closure* of A.

Similar to Example 5.37, we have the following example (Example 5.38).

Example 5.38 (Bartle and Sherbert p. 333 Question 16)**.** Let A and B be sets in $\mathbb{R}$. Show that we have

$$A \subseteq A^- \quad \text{and} \quad \left(A^-\right)^- = A^- \quad \text{and} \quad (A\cup B)^- = A^- \cup B^-.$$

Show that $(A\cap B)^- \subseteq A^- \cap B^-$, and give an example to show that the inclusion may be proper.

Solution. We first prove that $A \subseteq A^-$. By definition, the closure refers to the intersection of all closed sets containing A. Since A is included in this intersection, it follows that $A \subseteq A^-$.

We then prove that $(A^-)^- = A^-$. Recall that a set A is said to be closed if it contains all its limit points, i.e. $A = A^-$. The closure A^- of any set A is by definition closed. Taking the closure of A^- does not add any new points, so $(A^-)^- = A^-$.

We then prove that $(A\cup B)^- = A^- \cup B^-$. For the forward inclusion, let $x \in (A\cup B)^-$. By definition, every open neighbourhood U of x intersects $A\cup B$. Hence, for every open neighbourhood U containing x, either $U\cap A \neq \emptyset$ or $U\cap B \neq \emptyset$. This implies that x is either a point of A^- or B^-, so $x \in A^-$ or $x \in B^-$. Hence, $x \in A^- \cup B^-$. As for the reverse inclusion, let $x \in A^- \cup B^-$. Then, either $x \in A^-$ or $x \in B^-$. In either case, every open neighbourhood U of x intersects A (or B, respectively). Therefore, U intersects $A\cup B$. Hence, $x \in \overline{A\cup B}$. It follows that $(A\cup B)^- = A^- \cup B^-$.

Lastly, we prove that $(A\cap B)^- \subseteq A^-\cap B^-$. Let $x\in(A\cap B)^-$. Then, every open neighbourhood U of x intersects $A\cap B$, which means that there is some point $y\in U$ such that $y\in A$ and $y\in B$. Hence, U also intersects A and intersects B separately. Therefore, x is in both A^- and B^-, implying that $x\in A^-\cap B^-$. It follows that $(A\cap B)^-\subseteq A^-\cap B^-$.

However, the inclusion may be proper. To see why, let $A=(0,1)$ and $B=(1,2)$. Then, $A\cap B=\emptyset$ so $(A\cap B)-=\emptyset$ but $A^-=[0,1]$ and $B^-=[1,2]$, for which $A^-\cap B^-=\{1\}$. This shows that the inclusion can be proper. □

Example 5.39 (Bartle and Sherbert p. 333 Question 17)**.** Give an example of

$$\text{a set } A\subseteq\mathbb{R} \quad \text{such that} \quad A^\circ=\emptyset \text{ and } A^-=\mathbb{R}.$$

Solution. We claim that $A=\mathbb{Q}$ works. To see why, A° consists of all points where there is an open interval around them completely contained in A. However, any open interval in $\mathbb{R}$ contains irrational numbers. Thus, no open interval can lie entirely within $\mathbb{Q}$, implying that $\mathbb{Q}^\circ=\emptyset$.

We then prove that $\mathbb{Q}^-=\mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ (every real number is the limit of a sequence of rationals), the closure of $\mathbb{Q}$ is $\mathbb{R}$. □

Definition 5.18 (dense subset)**.** Let X be a topological space. We say that

$$A \text{ is dense in } X \quad \text{if and only if} \quad \overline{A}=X.$$

Proposition 5.8. $\operatorname{Int}(A)\subseteq A\subseteq\overline{A}$

Proposition 5.9. The following hold:

(i) $\operatorname{Int}(A)=A$ if and only if A is open

(ii) $\overline{A}=A$ if and only if A is closed

Theorem 5.6. Let A be a subset of a topological space X. Then,

$$x\in\overline{A} \quad \text{if and only if} \quad \text{every open set } U \text{ containing } x \text{ intersects } A.$$

Proof. We shall prove the contrapositive statement instead. That is,

$$x\notin\overline{A} \quad \text{if and only if} \quad \text{there exists an open set } U \text{ containing } x \text{ that does not intersect } A.$$

For the forward direction, suppose $x\notin\overline{A}$. Then, the set $X\setminus\overline{A}$ is an open set containing x that does not intersect A. For the reverse direction, suppose there exists an open set U containing x that does not intersect A. Then, $X\setminus U$ is a closed set containing A and $x\notin X\setminus U$. Hence, $\overline{A}\subseteq X\setminus U$, which implies that $x\notin\overline{A}$. □

Definition 5.19 (neighbourhood)**.** Let X be a topological space and let $x\in X$. A neighbourhood of x is an open set containing set. More generally,

$$\text{a neighbourhood of any subset } A\subseteq X \quad \text{is} \quad \text{an open set } U \text{ of } X \text{ such that } A\subseteq U.$$

Definition 5.20 (limit point). Let X be a topological space. Let $A \subseteq X$. A point $x \in X$ is a limit point of A if and only if every open neighbourhood of x intersects A in some point other than x itself. That is to say,

$$\text{for any open } U \subseteq X \text{ with } x \in U \quad \text{oen has } (U \setminus \{x\}) \cap A \neq \emptyset.$$

Let A' denote the set of all limit points of A.

Example 5.40. If $A = (0,1]$, then $A' = [0,1]$.

Example 5.41. If $B = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, then $B' = \{0\}$.

Example 5.42. If $C = \{0\} \cup (1,2)$, then $C' = [1,2]$.

Example 5.43. Other basic examples include $\mathbb{Q}' = \mathbb{R}$ and $\mathbb{N}' = \emptyset$.

Theorem 5.7. $\overline{A} = A \cup A'$

Proof. For the reverse inclusion, we have $A \subseteq \overline{A}$ by Proposition 5.8. If $x \in A'$, then every open neighbourhood of x intersects A, so $x \in \overline{A}$. Hence, $A' \subseteq \overline{A}$.

For the forward inclusion, suppose $x \in A$. Then, $x \in A \cup A'$. If $x \in \overline{A} \setminus A$, then every open neighbourhood U of x intersects A necessarily in a point different from x because $x \notin A$. So, $x \in A'$ and hence $x \in A \cup A'$. □

Corollary 5.4 (limit point criterion). A set A is closed if and only if it contains all its limit points. That is, $A' \subseteq A$.

Example 5.44 (Bartle and Sherbert p. 337 Question 10). Let $K \neq \emptyset$ be a compact set in $\mathbb{R}$. Show that $\inf K$ and $\sup K$ exist and belong to K.

Solution. Since K is a non-empty compact subset of $\mathbb{R}$, by the Heine-Borel theorem, K is closed and bounded. It suffices to prove that $\sup K$ exists and $\sup K \in K$ (a similar argument can be applied to $\inf K$). Because K is bounded, by the completeness property of $\mathbb{R}$, $\sup K$ exists.

We then prove that $\sup K \in K$. Let $\alpha = \sup K$. Suppose on the contrary that $\alpha \notin K$. Since K is closed, then it contains all of its limit points. By the definition of supremum, since $\alpha = \sup K$, then

$$\text{for every } \varepsilon \text{ there exists } x \in K \quad \text{such that} \quad \alpha - \varepsilon < x \leq \alpha.$$

As such, we can construct a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that

$$x_n > \alpha - \frac{1}{n}.$$

By compactness, every sequence x_n has a convergent subsequence whose limit is in K. In particular, by the construction of x_n, the limit must satisfy $x = \alpha$, which contradicts our earlier assumption that $\alpha \notin K$. The result follows. □

Example 5.45 (Bartle and Sherbert p. 337 Question 11). Let $K \neq \emptyset$ be compact in $\mathbb{R}$ and let $c \in \mathbb{R}$. Prove that there exists a point $a \in K$ such that

$$|c - a| = \inf\{|c - x| : x \in K\}.$$

Solution. Consider

$$f:\mathbb{R}\to\mathbb{R} \quad \text{defined by} \quad f(x)=|c-x|.$$

This function is continuous because the absolute value function and subtraction are continuous operations. Since $K\subseteq\mathbb{R}$ is compact, then K is closed and bounded by the Heine-Borel theorem. Moreover, the restriction of f to K is also continuous. For continuous functions on compact sets, they attain their minimum and maximum values by the extreme value theorem. So, there exists $a\in K$ such that

$$f(a)=\min\{f(x):x\in K\}.$$

This minimum value is exactly

$$|c-a|=\inf\{|c-x|:x\in K\},$$

and the result follows. □

Similar to Example 5.45, one can show that for any subset $K\neq\emptyset$ compact in $\mathbb{R}$ and $c\in\mathbb{R}$, there exists a point $b\in K$ such that

$$|c-b|=\sup\{|c-x|:x\in K\}.$$

Example 5.46 (Bartle and Sherbert p. 337 Question 14)**.** If K_1 and K_2 are disjoint non-empty compact sets, show that there exist $k_i\in K_i$ such that

$$0<|k_1-k_2|=\inf\{|x_1-x_2|:x_i\in K_i\}.$$

Solution. Let

$$d=\inf\{|x_1-x_2|:x_1\in K_1,x_2\in K_2\}.$$

Define

$$f:K_1\times K_2\to\text{codomain}_f \quad \text{where} \quad f(x_1,x_2)=|x_1-x_2|$$

Since K_1 and K_2 are compact, $K_1\times K_2$ is also compact. f is continuous, so it attains its minimum by the extreme value theorem. As such, there exist $k_1\in K_1$ and $k_2\in K_2$ such that

$$f(k_1,k_2)=|k_1-k_2|=d.$$

Since $K_1\cap K_2=\emptyset$, there is no point that is common to both sets, so $d>0$. The result follows. □

Example 5.47 (Bartle and Sherbert p. 337 Question 15)**.** Give an example of disjoint closed sets F_1,F_2 such that

$$\inf\{|x_1-x_2|:x_i\in F_i\}=0.$$

Solution. Let $F_1=\mathbb{N}$ which is closed in $\mathbb{R}$, and

$$F_2=\left\{n+\frac{1}{2n}:n\in\mathbb{N}\right\} \quad \text{which is also closed in } \mathbb{R}.$$

Note that $F_1\cap F_2=\emptyset$ but the infimum of the mentioned set is $\inf\left|\frac{1}{2n}\right|$ over all $n\in\mathbb{N}$, which is 0. □

Definition 5.21 (Hausdorff space)**.** A topological space X is said to be Hausdorff if and only if for each pair x_1,x_2 of distinct points of X, there exist disjoint open neighbourhoods U_1 and U_2 of x_1 and x_2 respectively (Figure 14).

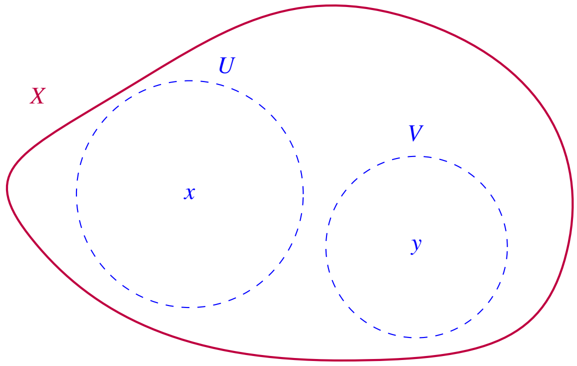


Figure 14: A Hausdorff space X

Example 5.48. Suppose the topology of X is trivial. Then,

$$X \text{ is Hausdorff} \quad \text{if and only if} \quad X \text{ is empty or } X \text{ is a singleton.}$$

Example 5.49. Suppose the topology of X is discrete. Then X is Hausdorff. To see why, given distinct points $x_1, x_2 \in X$, take $U_1 = \{x_1\}$ and $U_2 = \{x_2\}$, where both U_1 and U_2 are open in the discrete topology and $U_1 \cap U_2 = \emptyset$.

Example 5.50. Suppose X is a metric space (with the metric topology). Then, X is Hausdorff. To see why, given distinct points $x_1, x_2 \in X$, we have $r = d(x_1, x_2) > 0$. Then, take open balls $U_1 = B\left(x_1, \frac{r}{2}\right)$ and $U_2 = B\left(x_2, \frac{r}{2}\right)$ in X which are both open. Lastly, we need to show that $U_1 \cap U_2 = \emptyset$. Note that there exists $y \in U_1 \cap U_2$, then

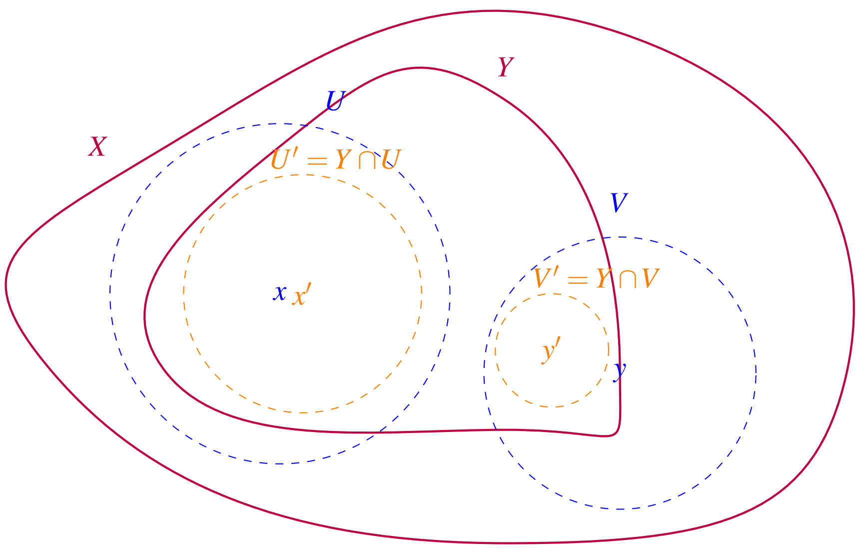
$$r = d(x_1, x_2) \leq d(x_1, y) + d(y, x_2) < \frac{r}{2} + \frac{r}{2} = r$$

which leads to a contradiction.

Proposition 5.10. A subspace of a Hausdorff space is also Hausdorff.

Proof. Let X be a Hausdorff space. Then, for all distinct points $x, y \in X$, we can construct two open balls U and V centred at x and y respectively such that $x \notin V$ and $y \notin U$. Suppose Y is a subspace of X. Then, consider two points $x', y' \in Y$. Consider the sets

$$U' = Y \cap U \quad \text{and} \quad V' = Y \cap V \quad \text{which are open in the subspace topology on } Y.$$



So,

$$U' \cap V' = U \cap V \cap Y \subseteq U \cap V = \emptyset.$$

So, there exists an open set $U' \subseteq Y$ such that if $x' \in U'$, then $x' \notin V'$. The same symmetric argument holds for y', i.e. there exists an open set $V' \subseteq Y$ such that if $y' \in V'$, then $y' \notin U'$. We conclude that Y is also Hausdorff. $\square$

Definition 5.22. Let X be a topological space. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. We say that $\{x_n\}_{n\in\mathbb{N}}$ converges to x in X and that x is a limit of the sequence if and only if

$$\text{for all open neighbourhoods } U \text{ of } x \quad \text{there exists } N \in \mathbb{N} \text{ such that for all } n \geq N \text{ we have } x_n \in U.$$

Example 5.51. If the topology of X is trivial, then for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, for any $x \in X$, one has $\{x_n\}_{n\in\mathbb{N}} \to x$ in X, so the limit of a sequence is not unique in a general topological space.

Example 5.52. Suppose the topology of X is discrete. Then, for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, for any $x \in X$, one has $\{x_n\}_{n\in\mathbb{N}} \to x$ in X if and only if $\{x_n\}_{n\in\mathbb{N}}$ is eventually constant of value x.

Example 5.53. Suppose X is a metric space. Then, for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, for any $x \in X$, one has $\{x_n\}_{n\in\mathbb{N}} \to x$ in X if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, one has $d(x_n, x) < \varepsilon$.

Example 5.54 (Munkres p. 171 Question 5)**.** Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B, respectively.

Solution. Let A and B be disjoint compact subspaces of X. Let $x \in A$. Since $x \notin B$ (because A and B are disjoint), there exist disjoint open sets U_x and V_x such that

$$x \in U_x \quad \text{and} \quad B \subseteq V_x.$$

The collection $\{U_x : x \in A\}$ is an open cover of A. Since A is compact, there exists a finite subcover $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ that covers A. That is,

$$A \subseteq U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}.$$

Define

$$U = U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}.$$

Then U is an open set containing A. Similarly, define

$$V = V_{x_1} \cap V_{x_2} \cap \cdots \cap V_{x_n}.$$

Since each V_{x_i} is open and contains B, the set V is an open set containing B. Moreover, $U \cap V = \emptyset$ because for each $1 \leq i \leq n$, $U_{x_i} \cap V_{x_i} = \emptyset$. The result follows. $\square$

Lemma 5.3. Let K be a compact set in a Hausdorff space X. Then, K is closed.

Proof. Fix $x \in X \setminus K$. Since X is Hausdorff, then for each $y \in K$, there exist disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Let $\{V_y : y \in K\}$ be an open cover of K. Since K is compact, then this open cover admits a finite subcover, say $\{V_y : y \in F\}$ for some finite subset of K. Let

$$U = \bigcap_{y\in F} U_y$$

which is an open neighbourhood of x disjoint from K. Since x was an arbitrary point of $X \setminus K$, then K is closed. $\square$

Example 5.55 (MA2108 AY24/25 Sem 2 Problem Set 5 Question 21)**.** Show that if

$$f : X \to Y \text{ is continuous} \quad \text{where} \quad X \text{ is compact and } Y \text{ is Hausdorff},$$

then f is a closed map (that is, f carries closed sets to closed sets).

Solution. Let $A \subseteq X$ be a closed set. Since X is compact, the closed subset A is also compact. Because f is continuous, the image $f(A)$ is compact in Y. Since Y is Hausdorff, then $f(A)$ is closed since every compact set in a Hausdorff space is closed (Lemma 5.3). Hence, $f(A)$ is closed in Y. This shows that f carries closed sets to closed sets, and thus f is a closed map. □

Theorem 5.8. Let X be a Hausdorff space and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then, $\{x_n\}_{n\in\mathbb{N}}$ converges to at most one point in x.

Proof. Suppose there exist distinct $x, x' \in X$ such that $x_n \to x$ and $x_n \to x'$. If $x \neq x'$ in X, because X is Hausdorff, then there exist U, U' open in X such that

$$x \in U \quad \text{and} \quad x' \in U' \quad \text{and} \quad U \cap U' = \emptyset.$$

Since $x_n \to x$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $x_n \in U$. Similarly, since $x_n \to x'$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $x_n \in U'$. Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we must have $x_n \in U \cap U'$. But this contradicts the fact that $U \cap U' = \emptyset$. So, our assumption that a sequence can converge to two distinct points must be false. □

Theorem 5.9 (Heine-Borel theorem)**.** Let $K \subseteq \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. Then,

$$K \text{ is compact} \quad \text{if and only if} \quad K \text{ is closed and bounded in } \mathbb{R}^n.$$

Proof. Suppose on the contrary that $I_0 = [a, b]$ is not compact. Then, there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of I_0 which does not reduce to a finite subcover. Consider the subintervals $I_0^L = \left[a, \frac{a+b}{2}\right]$ and $I_0^R = \left[\frac{a+b}{2}, b\right]$ of I_0. Then, $\{U_\alpha\}_{\alpha\in A}$ as an open cover of I_0^L and I_0^R, cannot both reduce to finite subcovers. This is because if

$$\text{there exist } A^L, A^R \subseteq A \quad \text{such that} \quad I_0^L \subseteq \bigcup_{\alpha\in A^L} U_\alpha \text{ and } I_0^R \subseteq \bigcup_{\alpha\in A^R} U_\alpha,$$

then $A_0 = A^L \cup A^R$ is finite and $\subseteq A$ such that

$$I_0 \subseteq \bigcup_{\alpha\in A_0} U_\alpha,$$

which is a contradiction. Denote by I_1 the subinterval (either I_0^L or I_0^R) that does not admit a finite subcover. Split I_1 into two halves, which are I_1^L and I_1^R. By the same reasoning, at least one of these halves, call it I_2, must also have the property that no finite subcollection of $\{U_\alpha\}_\alpha$ covers it. Repeating this process indefinitely produces a nested sequence of closed intervals

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots,$$

where the length of I_n is $|I_n| = \frac{b-a}{2^n}$. Note that

$$\lim_{n\to\infty} |I_n| = 0.$$

By the nested interval property in $\mathbb{R}$ (Theorem 2.11), the intersection

$$\bigcap_{n=0}^{\infty} I_n \quad \text{is non-empty.}$$

In fact, the intersection consists of exactly one point. Let

$$x_0 \in \bigcap_{n=0}^{\infty} I_n.$$

Since $\{U_\alpha\}_{\alpha \in A}$ is an open cover of I_0, there exists some index α_0 such that $x_0 \in U_{\alpha_0}$. Because U_{α_0} is open, there exists $\varepsilon > 0$ such that

$$(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq U_{\alpha_0}.$$

Since the lengths $|I_n|$ tend to zero, there exists $N \in \mathbb{N}$ such that the interval I_N is completely contained in the ε-neighbourhood of x_0. That is,

$$I_N \subseteq (x_0 - \varepsilon, x_0 + \varepsilon) \subseteq U_{\alpha_0}.$$

The interval I_N is a member of our nested sequence and, by construction, it was assumed to have no finite subcover by the members of $\{U_\alpha\}$. However, we have just shown that I_N is entirely contained in the single open set U_{α_0}. This means that the single set U_{α_0} alone covers I_N, providing a finite subcover for I_N. This contradicts our assumption that no finite subcover exists for I_N.

Thus, our original assumption that $[a,b]$ is not compact must be false. Therefore, every closed and bounded interval in $\mathbb{R}$ is compact.

The argument above establishes the compactness of closed and bounded intervals in $\mathbb{R}$. To extend this result to $\mathbb{R}^n$, observe that any closed and bounded subset $K \subset \mathbb{R}^n$ is contained within a closed n-dimensional box, that is, a Cartesian product of closed bounded intervals:

$$K \subseteq [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n].$$

Since the finite product of compact spaces is compact (a fact that follows from the Tychonoff theorem[†] or can be seen directly in $\mathbb{R}^n$ via a similar argument to the one-dimensional case), the box is compact. Moreover, K is closed as assumed, and a closed subset of a compact set is compact. Hence, K is compact. $\square$

Theorem 5.10 (topological generalisation of the extreme value theorem)**.** Let X be a non-empty compact topological space, and let

$$f : X \to \mathbb{R} \quad \text{be} \quad \text{a continuous function on } X.$$

Then, there exist $p, q \in X$ such that $f(p) = \sup f(X)$ and $f(q) = \inf f(X)$ in $\mathbb{R}$.

Proof. Since X is non-empty and compact, and f is continuous, then $f(X)$ is a non-empty compact subset of $\mathbb{R}$. By the Heine-Borel theorem (Theorem 5.9), $f(X)$ is non-empty, closed and bounded. Since $f(X)$ is non-empty and bounded, by the least upper bound property of $\mathbb{R}$, $\sup f(X)$ and $\inf f(X)$ exist in $\mathbb{R}$. Also, since $f(X)$ is closed, they belong to $f(X)$. $\square$

Theorem 5.11 (topological generalisation of the intermediate value theorem)**.** Let X be any connected topological space, and let

$$f : X \to \mathbb{R} \quad \text{be} \quad \text{a continuous function on } X.$$

Suppose $a, b \in X$ such that $f(a) \le f(b)$ in $\mathbb{R}$. Then, for all $t \in \mathbb{R}$ with $f(a) \le t \le f(b)$, there exists $p \in X$ such that $f(p) = t$ in $\mathbb{R}$.

[†] Will encounter in MA3209. Tychonoff's theorem states that the arbitrary product of compact spaces is also compact.

Proof. Since X is connected and f is continuous, then $f(X)$ is a connected subset of $\mathbb{R}$ by Theorem 5.1. Hence, $f(X)$ is convex, i.e. $f(a) \leq f(b)$ in $f(X)$, which implies $[f(a), f(b)] \subseteq f(X)$. $\square$